VISCOMETRIC MOTIONS OF GRANULAR MATERIALS

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LIST OF SYMBOLS

The following symbols are listed in their order of appearance in the thesis.

ν	volume distribution function
	material volume of grains in a distributed body
$\underline{v}$	velocity
ν_o	volume distribution at an initial state
J	the Jacobian, equivalent to det $\|F\|$, where $\|F\| = \left\|\frac{\partial x_i}{\partial X_\alpha}\right\|$
$\underline{q}$	heat flux vector
$\underline{\underline{T}}$	stress tensor
$\underline{b}$	body force vector
ε	specific internal energy
r	heat supply
η	specific entropy
θ	temperature
$\underline{\phi}$	entropy flux vector
ℓ	external equilibrated body force
k	equilibrated inertia
$\underline{h}$	equilibrated stress tensor
g	intrinsic equilibrated body force, acceleration due to gravity
γ	mass density of the grains
p	material pressure due to compressibility of grains
$\hat{p}$	pressure arising from distribution of grains

λ	bulk viscosity
μ	viscosity
$\underset{\sim}{n}$	unit normal vector
ψ	specific free energy
$\alpha_o, \beta_o, \alpha, \beta$	material coefficients of granular materials
$\underset{\approx}{D}$	rate of deformation tensor
$L = \sqrt{\frac{\beta}{\alpha}}\, \ell$	length ratio
$M = \frac{\gamma g \ell}{\beta}$	dimensionless parameter related to the external body forces
ℓ	characteristic length of the geometry being considered
K_1, K_2, K_3	functions used to describe the Reiner-Rivlin fluid
$\underset{\approx}{F}(s)$	history of the relative deformation gradient
$\mathscr{H}$	functional used in definition of simple fluids
t	the present time
τ	a particular time in the past
s	$s = t - \tau$
$\underset{\approx}{T}_E$	extra stress
$\underset{\sim}{T}^o$	nondissipative or equilibrium stress
$\underset{\sim}{X}$	material points
$\underset{\sim}{x}$	spatial points
$\underset{\sim}{\mathscr{X}}_t$	relative deformation function
$\underset{\approx}{F}_t(\tau)$	relative deformation gradient
$\underset{\approx}{R}(s)$	an arbitrary orthogonal tensor
$\underset{\approx}{I}$	unit tensor
$\underset{\approx}{M}$	tensor used to define viscometric motions
$\underset{\sim}{b}_{<i>}$	basis used to define viscometric motions

k	rate of shear
$\underset{\sim}{e}_{<i>}$	basis used to define curvilineal motions
e_i	magnitudes of basis vectors used to define curvilineal motions
$b_x, b_y, b_z, b_r, b_\theta, b_z$	body forces for Cartesian and polar coordinate systems
a, b	parameters used to define the yield point
Ω_0, Ω_1	angular velocities of cylinders
J_o	Bessel function of the first kind of zeroth order*
Y_o	Bessel function of the second kind of zeroth order
J_1	Bessel function of the first kind of first order
Y_1	Bessel function of the second kind of first order
I_o	modified Bessel function of the first kind of zero order
K_o	MacDonald's function of zero order
I_1	modified Bessel function of the first kind of first order
K_1	MacDonald's function of first order
x_{1m}	zeroes of J_1
V	body force potential

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*We have followed the definitions of these special functions as given by N. N. Lebedev [1972, 2].

SUMMARY

This work deals with the viscometric and curvilineal motions of incompressible Coulomb granular materials. In two of their papers* Goodman and Cowin have presented a theory for granular materials based on the introduction of a new independent scalar variable into the continuity equation, constitutive equations, and balance equations governing the motions of such materials. In their 1971 paper Goodman and Cowin solved two types of boundary value problems for the motions of granular materials using the linear Coulomb granular model based on their theory. In the 1972 paper they extended the constitutive equation to cover more general classes of materials.

The two solutions found in the 1971 paper are the only published solutions using the linear theory. Inasmuch as the theory shows promise of adequately representing the motions of granular materials under the proper circumstances, it would be beneficial to obtain solutions to other problems of the type for which exact or approximate solutions for Navier-Stokes fluids are known to exist. This is the ultimate objective of the present thesis.

*Goodman and Cowin [1971, 1] and Goodman and Cowin [1972, 1].

The thesis consists of six chapters. The first chapter considers the theory of granular materials as presented by Goodman and Cowin in their papers and describes the restrictions and modifications made for its use in this thesis. The conclusion of Chapter I outlines the objectives of the thesis. Chapter II is devoted to a fuller presentation of the constitutive equation as defined for granular materials, putting it in a form most easily used for applying known results for viscometric and curvilineal motions. Chapter III derives the general results for the form of the constitutive equations for viscometric, and, more specifically, for curvilineal motions. Chapter IV considers specific problem solutions for the "modified" Navier-Stokes equations obtained by insertion of the constitutive equation derived in Chapter III into the linear momentum balance equation. Solutions are obtained for channel flow at an arbitrary angle, flow between concentric rotating cylinders, longitudinal motion between concentric vertical cylinders, and Poiseuille flow through a circular pipe. Chapter V considers torsional flow within a circular cylinder bounded by two rotating discs, which is not a viscometric flow as defined in the thesis. Chapter VI summarizes the results and presents recommendations. Finally, there are four appendices. Appendix A derives conditions for the steady universal motions of incompressible granular materials. Appendix B considers the definition of the regions of equilibrium and nonequilibrium as used in this work and the yield condition employed as a result of this definition. Appendix C is a discussion of the nonuniqueness of the solutions in general. Appendix D derives a result for incompressible Coulomb granular materials implied by the statement of the balance of equilibrated force.

Thus, the stated objective of this thesis is to obtain solutions for the viscometric motions of granular materials. The derivation of results valid for viscometric and curvilineal motions of granular materials is completed. In regard to the application of these results to specific boundary value problems, some exact solutions are obtained. For other problems, solutions are found by changing the form of the "modified" Navier-Stokes equations, in most cases by considering the inertia term to be negligible. The resulting solutions seem to match well with physical experience for "slow" motions. Finally, conditions for the steady universal motions of incompressible Coulomb granular materials are derived, and specializations for certain types of velocity fields are determined.

CHAPTER I

INTRODUCTION TO THE THEORY OF GRANULAR MATERIALS

The Theory of Goodman and Cowin

A theory of granular materials was recently proposed by Goodman and Cowin [1971, 1] and [1972, 1] which differs from previous theories by the introduction of an independent scalar variable called the volume distribution function. The granular nature of the material is reflected in this function and in the stress representation. This new variable can be interpreted as the ratio of the material volume occupied by the grains to the total spatial volume instantaneously containing the material volume. The difference of these two volumes represents the amount of void space between the grains.

For incompressible motions the volume distribution function ν has the property that $0 \leq \nu(\underset{\sim}{x},t) \leq 1$ for all $\underset{\sim}{x} \in B$ and all t, where B is some distributed body of granular material. The volume $\mathcal{V}$ of grains in the distributed body is therefore

$$\mathcal{V} = \int_B \nu dv. \tag{1.1}$$

In their work, Goodman and Cowin present constitutive equations and derive through kinematic, dynamic, and thermodynamic considerations the equations of continuity and the equations of motion for a granular material. In this work we will consider only the linear constitutive equations and the equation of motion derived from them. Furthermore, we will consider only isochoric motions for which

$$\operatorname{div} \underset{\sim}{v} = 0. \tag{1.2}$$

The continuity equation is derived through kinematic arguments. It may be written as

$$\dot{\gamma}\nu + \gamma(\dot{\nu} + \nu \operatorname{div} \underset{\sim}{v}) = 0. \tag{1.3}$$

The following balance equations are assumed by Goodman and Cowin:

(1) The Balance of Energy

$$\frac{d}{dt}\int_{p_t} \gamma\nu(\varepsilon + \frac{1}{2}\underset{\sim}{v}\cdot\underset{\sim}{v} + \frac{1}{2}k\dot{\nu}\dot{\nu})dv = \int_{\partial p_t} (\underset{=}{T}^T\cdot\underset{\sim}{v} + \underset{\sim}{h}\dot{\nu} - \underset{\sim}{q})\cdot\underset{\sim}{n}dA$$

$$+ \int_{p_t} \gamma\nu(\underset{\sim}{b}\cdot\underset{\sim}{v} + \ell\dot{\nu} + r)dv, \tag{1.4}$$

(2) The Balance of Equilibrated Force

$$\frac{d}{dt}\int_{p_t} \gamma\nu k\dot{\nu}dv = \int_{\partial p_t} \underset{\sim}{h}\cdot\underset{\sim}{n}\, dA + \int_{p_t} \gamma\nu(\ell+g)dv, \tag{1.5}$$

(3) The Balance of Equilibrated Inertia

$$\frac{d}{dt} \int_{p_t} \gamma\nu k \, dv = 0, \tag{1.6}$$

and (4) The Clausius-Duhem Inequality

$$\frac{d}{dt} \int_{p_t} \gamma\nu\eta \, dv \geq \int_{\partial p_t} \underset{\sim}{\phi} \cdot \underset{\sim}{n} \, dA + \int_{p_t} \gamma\nu \frac{r}{\theta} \, dv. \tag{1.7}$$

Goodman and Cowin point out the difference between the energy and entropy statements given above and the classical statements. They state that the $\dot{\nu}$ terms appear because ν is independent of both the motion and the temperature.

The conservation of linear momentum for granular materials may be written as

$$\operatorname{div} \underset{\approx}{T} + \gamma\nu \underset{\sim}{b} = \gamma\nu \dot{\underset{\sim}{v}}. \tag{1.8}$$

From the conservation of angular momentum we obtain

$$\underset{\approx}{T} = \underset{\approx}{T}^T. \tag{1.9}$$

Goodman and Cowin [1972, 1] obtain the following field equation from the balance of equilibrated force:

$$\gamma\nu k\ddot{\nu} = \text{div}\ \underset{\sim}{h} + \gamma\nu(\ell+g)\ . \tag{1.10}$$

In Appendix D we will derive a result for isochoric motions of granular materials which follows directly from (1.10).

In obtaining their representation for the stress in granular materials, Goodman and Cowin [1971, 1] define three quantities as follows:

$$p = \gamma^2\nu\ \frac{\partial\psi}{\partial\gamma}\ , \tag{1.11}$$

$$\hat{p} = \gamma\nu^2\ \frac{\partial\psi}{\partial\nu}\ , \tag{1.12}$$

$$\underset{\sim}{h} = \gamma\nu\ \frac{\partial\psi}{\partial(\text{grad}\ \nu)}\ . \tag{1.13}$$

These three quantities are related by the equation

$$\hat{p}-p = \nu\ \text{div}\ \underset{\sim}{h}. \tag{1.14}$$

They interpret the pressure p as a material pressure corresponding to the compressibility of the grains.

$\hat{p}$ is considered to be a pressure arising from the distribution of grains given by the granular distribution function. The vector $\underset{\sim}{h}$, called the <u>equilibrated stress</u> vector, represents a system of forces between the grains which is in self-equilibrium.

These definitions result in the definition of the equilibrium stress $\underline{\underline{T}}^{o}$, which is given by

$$\underline{\underline{T}}^{o} = -p\underline{\underline{I}} - \underline{h} \otimes \operatorname{grad} \nu, \tag{1.15}$$

which is equivalent to

$$\underline{\underline{T}}^{o} = -\hat{p}\underline{\underline{I}} + \nu(\operatorname{div} \underline{h})\underline{\underline{I}} - \underline{h} \otimes \operatorname{grad} \nu. \tag{1.16}$$

We define the extra-stress for the granular material in the same manner as for a Navier-Stokes fluid:

$$\underline{\underline{T}}_E = \lambda(\mathrm{tr}\underline{\underline{D}})\underline{\underline{I}} + 2\mu\underline{\underline{D}}. \tag{1.17}$$

However, since we are considering only isochoric motions, $\underline{\underline{T}}_E$ will reduce to

$$\underline{\underline{T}}_E = 2\mu\underline{\underline{D}}. \tag{1.18}$$

Goodman and Cowin derive a constitutive equation for what they call a Coulomb granular material.* By employing a Taylor series expansion of the specific free energy per unit volume $\gamma\nu\psi$ they obtain the following expressions for p, $\hat{p}$, and $\underline{h}$:

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*Jenkins [1975, 1] has found motivation for considering an alternative expression for the specific free energy function.

$$p = \frac{\gamma \partial \alpha_o}{\partial \gamma} - \alpha_o + (\frac{\gamma \partial \alpha}{\partial \gamma} - \alpha) \text{grad } \nu \cdot \text{grad } \nu, \tag{1.19}$$

$$\hat{p} = - \beta_o + \beta \nu^2 - \alpha \text{ grad } \nu \cdot \text{grad } \nu, \tag{1.20}$$

and $$\underline{h} = 2\alpha \text{grad } \nu. \tag{1.21}$$

Goodman and Cowin point out that in the equilibrium regions of a granular material, the normal and shear stress at any point will be related by

$$S^2 + (T-t)^2 = s^2, \tag{1.22}$$

where $s = \alpha \text{ grad } \nu \cdot \text{grad } \nu$ and $t = - p - \alpha \text{ grad } \nu \cdot \text{grad } \nu$. The normal stress T is considered to be acting across an arbitrary fixed plane in the equilibrium region, and the shear stress S is acting in the same plane.

The general constitutive equation for the cohesionless ($\beta_o = 0$) Coulomb granular material is of the form

$$\underline{\underline{T}} = (-\beta\nu^2 + \alpha \text{ grad } \nu \cdot \text{grad } \nu + 2\nu \text{ div } (\alpha \text{ grad } \nu))\underline{\underline{I}} - 2\alpha \text{ grad } \nu \otimes \text{grad } \nu + 2\mu \underline{\underline{D}} \tag{1.23}$$

in the regions of nonequilibrium for isochoric motions, which is modified to

$$S^2 + (T-t)^2 = s^2 \tag{1.24}$$

in the regions of equilibrium.*

The distinctions between the regions of equilibrium and nonequilibrium are important. We define our regions of nonequilibrium to be finite regions for which $\underline{\underline{D}} \neq \underline{\underline{0}}$. Clearly, then, the regions of equilibrium are those regions where $\underline{\underline{D}} = \underline{\underline{0}}$. This is a slightly different interpretation than that used by Goodman and Cowin. They employ the Coulomb yield condition as a boundary condition for the separation of the two regions in the solutions to the two granular flow problems considered in [1971, 1]. We will use the alternative condition that $\underline{\underline{D}} = \underline{\underline{0}}$ at the interface between regions of equilibrium and nonequilibrium.

If the dynamic stress relation is inserted into the balance of linear momentum, we obtain

$$\gamma\nu\dot{\underline{v}} = -2\beta\nu \text{ grad } \nu + 2\alpha\nu \text{ grad}(\nabla^2\nu) + \mu\nabla^2\underline{v} + \gamma\nu\underline{b} \tag{1.25}$$

for isochoric motions. We will call this equation the "modified" Navier-Stokes equation.

Finally, Goodman and Cowin indicate [1971, 1] that two new dimensionless parameters are of interest in this

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*Jenkins [1975, 1] derives an alternative condition satisfied in the regions of equilibrium.

granular theory. They are the length ratio*

$$L = \sqrt{\frac{\beta}{\alpha}}\,\ell \quad \text{and} \quad M = \frac{\gamma_g \ell}{\beta}\,, \qquad (126a,b)$$

where ℓ is a characteristic length of the geometry of the system being considered.

The Objective of the Thesis

Thus we have a very brief exposition of the theory of granular materials as proposed by Goodman and Cowin. The objective of this thesis as a whole will be the determination of the form of the linear constitutive equation of granular materials for viscometric motions. We will continue these results to the case of curvilineal flows, which form a broad class of motions comprising many of the flows for which solutions have been obtained in the case of Navier-Stokes fluids. We will obtain exact solutions for many specific cases of these motions. We shall find for many of the flows considered that we can obtain no exact solutions using the techniques considered in this thesis. In such cases we will attempt to obtain inexact solutions by making what we claim are reasonable assumptions under the appropriate conditions. We will state the appropriate physical conditions for which our inexact solutions should be valid.

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*For a discussion of a similar material characteristic length parameter used in the theory of a polar fluid, see Cowin, [1967, 1].

After a number of curvilineal motions are considered, we shall look at a motion which is not viscometric as we have defined it. This is a torsional flow bounded by a stationary vertical cylinder and two rotating discs. We will obtain solutions for this problem by neglecting the inertia term which occurs in one of the "modified" Navier-Stokes equations. This will allow us to obtain an inexact solution for this case as well.

Finally, general conditions required for the steady universal motions of incompressible Coulomb granular materials will be derived. Other results will be obtained by specifying various types of velocity fields considered.

In the following chapter we will attempt to formalize the concept of stress. We will compare our constitutive equation with that obtained for more general materials. However, it must be pointed out that in this work we are considering in this theory of granular materials a constitutive equation for which the extra stress in dependent only on the instantaneous value of the relative deformation gradient.

CHAPTER II

THE CONSTITUTIVE EQUATION

Historical Discussion

The most famous constitutive assumption concerning the motions of viscous fluids was that made by Stokes in 1845. He argued that the stress for such fluids should take the form

$$\underline{\underline{T}} = -p\underline{\underline{I}} + f(\underline{\underline{D}}), \tag{2.1}$$

the stress being a function of the hydrostatic pressure and having a functional dependence on the symmetric part of the velocity gradient. When he undertook the study of flows in the special case where

$$f(\underline{\underline{D}}) = 2\mu\underline{\underline{D}}, \tag{2.2}$$

he initiated the development of the theory of the familiar Navier-Stokes fluid. However, flow effects not explained by this theory of linearly viscous fluids were noticed by investigators, and it became evident that a constitutive equation of a different form was required for some fluids. About 30 years ago Reiner [1945, 1], Rivlin [1947, 1] and others began considering other possible forms of the constitutive equation. The stress was assumed to be represented by the more general form

$$p\underline{\underline{I}} + \underline{\underline{T}} = K_0\underline{\underline{I}} + K_1\underline{\underline{D}} + K_2\underline{\underline{D}}^2, \tag{2.3}$$

where K_0, K_1, and K_2 are scalar valued functions of the three principal invariants of $\underline{\underline{D}}$. These concepts were revised and expanded over a period of about 15 years until the concept of a simple fluid was perfected. The following definition for an incompressible simple fluid was obtained:

$$\det \underline{\underline{F}}(s) = 1 \tag{2.4}$$

and

$$\underline{\underline{T}}_E = \mathop{\mathscr{H}}_{s=0}^{\infty} (\underline{\underline{F}}(s)), \tag{2.5}$$

where $\underline{\underline{F}}(s)$ is the history of the relative deformation gradient (which depends on the choice of the material point $\underline{X}$ and the time t). We have τ = t-s, where t is the present time and τ is a time in the past. s is therefore a measure of how far back into the past history of the deformation of the material we are going. Equation (2.4) states that only isochoric motions are permitted. Equation (2.5) states that the extra stress is determined by the history of the relative deformation gradient, where $\mathscr{H}$ is some functional which is defined for a particular fluid.

The Extension of the Stress Concept

Goodman and Cowin's theory extends the concept of stress in another manner, for a different type of material. They assume that the extra stress which represents the dynamical part of the constitutive equation to be of a form that is identical to that for a Navier-Stokes fluid:

$$\underline{\underline{T}}_E = (\lambda+\mu) \ \text{tr}\underline{\underline{D}} + 2\mu\underline{\underline{D}} \tag{2.6}$$

The definition of extra stress was largely one of convenience, since it allowed us to look specifically at only that part of the stress tensor arising due to dynamic effects. In their theory of granular materials, Goodman and Cowin find it convenient to define another part of the stress tensor, which they call the equilibrium or nondissipative part:

$$\underline{\underline{T}}^{o} = -p\underline{\underline{I}} - \underline{h} \otimes \text{grad } \nu. \tag{2.7}$$

This may also be written as

$$\underline{\underline{T}}^{o} = -\hat{p}\underline{\underline{I}} + (\nu \text{ div } \underline{h})\underline{\underline{I}} - \underline{h} \otimes \text{grad } \nu. \tag{2.8}$$

In Chapter I we stated that in the case of the linear theory of Coulomb granular materials employed by Goodman and Cowin, the representations for $\hat{p}$ and $\underline{h}$ take on the simple forms:

$$\hat{p} = -\beta_o + \beta\nu^2 - \alpha \text{ grad } \nu \cdot \text{grad } \nu \tag{2.9}$$

and

$$\underline{h} = 2\,\alpha \text{ grad } \nu, \tag{2.10}$$

where β_o, β and α are positive material coefficients which can in general depend on ν, γ and the temperature θ, but which we take to be constants for our purposes. Note that the definition of the equilibrium part of the stress implies that the equilibrium stress is a function solely of the newly introduced kinematic variable ν and is completely independent of the dynamic effects. It is also evident from (2.7) that the equilibrium stress for the Coulomb granular material is assumed to be dependent only on the magnitude of ν at the present time t and not on the past history of ν.

In this work we will be considering the motions of incompressible Coulomb granular materials. We limit ourselves to the <u>nondilitant</u> materials, which are granular materials capable only of isochoric motions. Goodman and Cowin also consider a different constraint on the kinematical nature of the motions considered. They define motions with <u>incompressible distributed volume</u> by the continuity equation

$$\dot{\nu} + \nu \text{div } \underline{v} = 0. \tag{2.11}$$

The continuity equation for isochoric motions reduces to

$$\dot{\nu} = 0. \tag{2.12}$$

In words, the incompressible Coulomb granular material is defined as follows:

(1) only isochoric motions are permitted,

(2) the continuity equation for isochoric motions must be satisfied, and

(3) the stress tensor is composed of two parts. One part is simply the extra stress for incompressible Navier-Stokes fluids. The second part is the nondissipative part which is dependent on the volume distribution function. The extra stress describes the dynamic stress effects, while the nondissipative stress is dependent on the distribution of granules.

Mathematically we have

$$(1) \qquad \operatorname{div} \underline{v} = 0, \tag{2.13}$$

$$(2) \qquad \dot{\nu} = \frac{\partial \nu}{\partial t} + \underline{v} \cdot \operatorname{grad} \nu = 0, \tag{2.14}$$

and

$$(3) \qquad \underline{\underline{T}} = \underline{\underline{T}}_E + \underline{\underline{T}}^{\circ}, \tag{2.15}$$

where

$$\underline{\underline{T}}_E = 2\mu \underline{\underline{D}} \tag{2.16}$$

and $$\underline{\underline{T}}^{o} = -\hat{p}\underline{\underline{I}} + \nu(\text{div } \underline{h})\underline{\underline{I}} - \underline{h} \otimes \text{grad } \nu, \tag{2.17}$$

$\hat{p}$ and $\underline{h}$ as defined earlier.

Any proposed constitutive equation must satisfy the principle of material objectivity, which is defined by Coleman, Markovitz, and Noll [1966, 1] as follows: "If a given process is compatible with a constitutive equation, then all processes obtained from the given process by changes of frame must also be compatible with the same constitutive equation." This principle restricts the possible candidates of functionals satisfying equation (2.5). For the incompressible Coulomb granular material, since the extra stress is the same as that for a Navier-Stokes fluid, the $\underline{\underline{T}}_E$ part of the total stress satisfies material objectivity. The equilibrium stress as defined by Goodman and Cowin satisfies material objectivity under restrictions outlined in [1972, 1]. Essentially, material objectivity implies that $\underline{\underline{T}}^{o}$ and $\underline{h}$ be isotropic functions of their vector and tensor arguments grad ν, grad θ and $\underline{\underline{D}}$. Isotropy in this sense means that the response of $\underline{\underline{T}}^{o}$ and $\underline{h}$ to grad ν, grad θ and $\underline{\underline{D}}$ is the same in all directions. The Coulomb granular material satisfies all the restrictions on both $\underline{\underline{T}}_E$ and $\underline{\underline{T}}^{o}$. If the two parts of a constitutive equation each separately satisfy material objectivity, then the sum also satisfies material objectivity.

The next natural step to take for the generalization of the theory of granular materials is the construction of a model of a simple granular material. The extra stress response would in general be a functional of the history of the relative deformation gradient as well as ν and grad ν. This is beyond the scope of the present work.

CHAPTER III

VISCOMETRIC AND CURVILINEAL MOTIONS OF GRANULAR MATERIALS

Specialization of Other Results

In this chapter we will apply some general results on viscometric and curvilineal motions described in Coleman, Markovitz, and Noll [1966, 1] and in Truesdell and Noll [1965, 1] to the case of incompressible Coulomb granular materials. These general results will then be used to consider specific types of curvilineal motions for incompressible Coulomb granular materials in later chapters.

All the results outlined here were obtained in the above works for incompressible simple fluids. The constitutive equation has the very general form

$$\underline{\underline{T}}_E = \mathop{\mathscr{H}}_{s=0}^{\infty} (\underline{\underline{F}}(s)), \tag{3.1}$$

as stated in Chapter II. For facilitating the comparison of results obtained for simple fluids and Coulomb materials, we note that we can apply all of the results for simple fluids to the Coulomb materials by setting

$$\mathop{\mathscr{H}}_{s=0}^{\infty} (\underline{\underline{F}}(s)) = -\mu \frac{d}{ds} (\underline{\underline{F}}(s) + \underline{\underline{F}}(s)^T)\Big|_{s=0} . \tag{3.2}$$

For the Coulomb material, the extra stress is dependent only on the deformation gradient at the present time. Since

$$\frac{d}{ds}\left(\underset{=}{F}(s)\right)\Big|_{s=o} = -\mathrm{grad}\ \underset{\sim}{v} \quad \text{and} \quad \frac{d}{ds}\left(\underset{=}{F}(s)^T\right)\Big|_{s=o} = -(\mathrm{grad}\ \underset{\sim}{v})^T \tag{3.3}$$

we have

$$\underset{s=o}{\overset{\infty}{\mathscr{H}}}\left(\underset{=}{F}(s)\right) = \mu(\mathrm{grad}\ \underset{\sim}{v} + (\mathrm{grad}\ \underset{\sim}{v})^T). \tag{3.4}$$

However,

$$\underset{=}{D} = \frac{1}{2}\ (\mathrm{grad}\ \underset{\sim}{v} + (\mathrm{grad}\ \underset{\sim}{v})^T), \tag{3.5}$$

so our extra stress reduces to

$$\underset{=}{T}_E = 2\mu\underset{=}{D}, \tag{3.6}$$

which is identical to that for isochoric motions of Navier-Stokes fluids.

Viscometric Flows of Coulomb Granular Materials

Let us suppose that the material points $\underset{\sim}{X}$ of a Coulomb granular material in the regions of nonequilibrium are given by

$$\underline{X} = \underline{\mathscr{X}}_t\ (\underline{x}, \tau). \tag{3.7}$$

$\underline{\mathscr{X}}_t(\underline{x}, \tau)$ is the relative deformation function. We define the relative deformation gradient by the equation

$$\underline{\underline{F}}_t(\tau) = \nabla \underline{\mathscr{X}}_t\ (\underline{x}, \tau), \tag{3.8}$$

where $\underline{x}$ is the spatial position at the present time t of the material point we label as X. We further define

$$\underline{\underline{F}}(s) = \underline{\underline{F}}_t(t-s), \quad s \geq 0 \tag{3.9}$$

as the history of the relative deformation gradient.

Suppose further that the history of the relative deformation gradient for all $\underline{X}$,t is given by

$$\underline{\underline{F}}(s) = \underline{\underline{R}}(s)\ (\underline{\underline{I}} - s\underline{\underline{M}}), \tag{3.10}$$

where $\underline{\underline{R}}(s)$ is an orthogonal tensor for each s, with

$$\underline{\underline{R}}(0) = \underline{\underline{I}}. \tag{3.11}$$

The flow is said to be <u>viscometric</u> if $\underline{\underline{M}}$ is of the form

$$\underline{\underline{M}} = \begin{vmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{3.12}$$

with respect to some basis $\underline{b}\langle i\rangle$ of a right-handed orthogonal coordinate system. In general we have

$$\begin{aligned} k &= k(\underline{X},t) \\ \underline{\underline{R}}(s) &= \underline{\underline{R}}(s)\ (\underline{X},t) \\ \text{and}\qquad \underline{b}\langle i\rangle &= \underline{b}\langle i\rangle\ (\underline{X},t) \end{aligned} \tag{3.13}$$

This definition holds for both simple fluids and Coulomb granular materials. k is called the "rate of shear."

Recall that we are considering only the case where

$$\mathop{\mathscr{H}}\limits_{s=0}^{\infty} (\underline{\underline{F}}(s)) = -\mu \frac{d}{ds} (\underline{\underline{F}}(s) + \underline{\underline{F}}(s)^T)\Big|_{s=0} . \tag{3.14}$$

Thus

$$2\mu\underline{\underline{D}} = -\mu \frac{d}{ds} (\underline{\underline{F}}(s) + \underline{\underline{F}}(s)^T)\Big|_{s=0} . \tag{3.15}$$

For viscometric motions

$$\underline{\underline{F}}(s) = \underline{\underline{R}}(s)(\underline{\underline{I}}-s\underline{\underline{M}}), \tag{3.16}$$

so

$$\frac{d}{ds}\left(\underline{\underline{F}}(s)\right)\Big|_{s=0} = \underline{\underline{R}}(0)\ (-\underline{\underline{M}}) = -\underline{\underline{M}} \tag{3.17}$$

and

$$\frac{d}{ds}\left(\underline{\underline{F}}(s)^T\right)\Big|_{s=0} = -\underline{\underline{M}}^T(\underline{\underline{R}}^T(0)) = -\underline{\underline{M}}^T, \tag{3.18}$$

where we have

$$\left.\frac{d\underline{\underline{R}}(s)}{ds}\right|_{s=0} = 0. \tag{3.19}$$

Thus

$$2\mu\underline{\underline{D}} = \mu(\underline{\underline{M}} + \underline{\underline{M}}^T) = \mu \begin{vmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} . \tag{3.20}$$

Therefore the stress in the regions of nonequilibrium for a viscometric motion of a Coulomb granular material is

$$\underline{\underline{T}} = \mu \begin{vmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + (\beta_o - \beta\nu^2 + \alpha \text{ grad } \nu \cdot \text{grad } \nu + 2\alpha\nu\nabla^2\nu)\underline{\underline{I}} - 2\alpha \text{ grad } \nu \otimes \text{grad } \nu. \tag{3.21}$$

$\underline{\underline{T}}$ in the regions of equilibrium is given as in (1.24).

Now let

$$\text{grad } \nu = \frac{\partial \nu}{\partial x^1} \underline{b}_1 + \frac{\partial \nu}{\partial x^2} \underline{b}_2 + \frac{\partial \nu}{\partial x^3} \underline{b}_3, \tag{3.22}$$

with x^1, x^2, x^3 the orthogonal coordinates for a coordinate system defined by $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$, the basis for which $\underline{\underline{M}}$ is of the required form. Then

$$\text{grad } \nu \otimes \text{grad } \nu = \begin{vmatrix} \left(\frac{\partial \nu}{\partial x^1}\right)^2 & \frac{\partial \nu}{\partial x^1}\frac{\partial \nu}{\partial x^2} & \frac{\partial \nu}{\partial x^1}\frac{\partial \nu}{\partial x^3} \\ \frac{\partial \nu}{\partial x^1}\frac{\partial \nu}{\partial x^2} & \left(\frac{\partial \nu}{\partial x^2}\right)^2 & \frac{\partial \nu}{\partial x^2}\frac{\partial \nu}{\partial x^3} \\ \frac{\partial \nu}{\partial x^1}\frac{\partial \nu}{\partial x^3} \; \frac{\partial \nu}{\partial x^2}\frac{\partial \nu}{\partial x^3} & & \left(\frac{\partial \nu}{\partial x^3}\right)^2 \end{vmatrix} . \tag{3.23}$$

The general form of the stress in the regions of nonequilibrium for viscometric motions may then be written as

$$\underline{\underline{T}} = \mu \begin{vmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + (\beta_o - \beta\nu^2 + \alpha \text{ grad } \nu \cdot \text{grad } \nu + 2\alpha\nu\nabla^2\nu)\underline{\underline{I}} + \tag{3.24}$$

$$-2\alpha \begin{vmatrix} (\frac{\partial \nu}{\partial x^1})^2 & \frac{\partial \nu}{\partial x^1}\frac{\partial \nu}{\partial x^2} & \frac{\partial \nu}{\partial x^1}\frac{\partial \nu}{\partial x^3} \\ \frac{\partial \nu}{\partial x^1}\frac{\partial \nu}{\partial x^2} & (\frac{\partial \nu}{\partial x^2})^2 & \frac{\partial \nu}{\partial x^2}\frac{\partial \nu}{\partial x^3} \\ \frac{\partial \nu}{\partial x^1}\frac{\partial \nu}{\partial x^3} & \frac{\partial \nu}{\partial x^2}\frac{\partial \nu}{\partial x^3} & (\frac{\partial \nu}{\partial x^3})^2 \end{vmatrix} . \quad (3.25)$$

k is specified for a particular type of viscometric motion.

<u>Curvilineal Flows of Coulomb Granular Materials</u>

We now consider an important class of flows which are contained within the more general category of viscometric motions. <u>Curvilineal flows</u> are motions for which the velocity field $\underline{v} = \underline{v}(\hat{\underline{x}})$ has the form

$$v^1 = 0, \quad v^2 = v^2(\hat{x}^1), \quad v^3 = v^3(\hat{x}^1) \quad (3.26)$$

in an orthogonal coordinate system $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$ for which the magnitudes e_i of the natural basis vectors are constant along curves $\underline{y} = \underline{y}(s)$ which have the parametric representation

$$y^1 = \hat{x}^1, \quad y^2 = \hat{x}^2 - sv^2(\hat{x}^1), \quad y^3 = \hat{x}^3 - sv^3(\hat{x}^1) . \quad (3.27)$$

It has been shown* that in regard to curvilineal flows,

---

*For a proof of the results, see Coleman, Markovitz and Noll, [1966, 1, §13], or Truesdell and Noll, [1965, 1, pp. 432-435].

(1) every curvilineal flow is viscometric,

(2) the rate of shear k is given by

$$k = \frac{1}{e_1}\sqrt{(v^2)^1\; e_2{}^2 + (v^3)^1\; e_3{}^2}\,, \tag{3.28}$$

where

$$(v^2)^1 = \frac{dv^2(\hat{x}^1)}{d\hat{x}^1} \quad \text{and} \quad (v^3)^1 = \frac{dv^3(\hat{x}^1)}{d\hat{x}^1}, \tag{3.29}$$

and (3) the basis $\underline{b}<i>$ with respect to which $\underline{\underline{M}}$ is of the required form is related to the basis with which we defined curvilineal flows by

$$\begin{aligned} \underline{b}<1> &= \underline{e}<1>, \\ \underline{b}<2> &= m\underline{e}<2> + n\underline{e}<3>, \\ \underline{b}<3> &= -n\underline{e}<2> + m\underline{e}<3>, \end{aligned} \tag{3.30}$$

where $\underline{e}<i>$ is the normalized natural basis for our coordinate system $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$, and

$$m = \frac{(v^2)^1}{k}\,\frac{e_2}{e_1} \quad \text{and} \quad n = \frac{(v^3)^1}{k}\,\frac{e_3}{e_1}\,. \tag{3.31}$$

Furthermore, $\underline{\underline{M}}$ is given with respect to $\underline{e}<i>$ by

$$\underline{\underline{M}} = \begin{vmatrix} 0 & 0 & 0 \\ \frac{e_2}{e_1}(v^2)_1 & 0 & 0 \\ \frac{e_3}{e_1}(v^3)_1 & 0 & 0 \end{vmatrix} . \tag{3.32}$$

Once again, all of the above results can be applied to the case of Coulomb granular materials.

The Form of the Stress Tensor for Curvilineal Motions of Coulomb Granular Materials

All curvilineal flows are viscometric, so that the stress in the equilibrium and nonequilibrium regions obeys the results already derived for the viscometric case. However, we will be concerned with the physical components $T_{<ij>}$ of the stress with respect to the coordinate system used for defining curvilineal flow, which are the components of $\underline{\underline{T}}$ relative to $\underline{e}\langle i\rangle$. These can be expressed as

$$T_{<ij>} = \underline{e}\langle i\rangle \cdot \underline{\underline{T}}\, \underline{e}\langle j\rangle . \tag{3.33}$$

We use the transformation rule

$$\underline{e}\langle i\rangle \cdot \underline{\underline{T}}\, \underline{e}\langle j\rangle = \sum_{\ell=1}^{3} \sum_{k=1}^{3} (\underline{e}\langle i\rangle \cdot \underline{b}\langle k\rangle)(\underline{b}\langle k\rangle \cdot \underline{\underline{T}}\underline{b}\langle \ell\rangle)(\underline{b}\langle \ell\rangle \cdot \underline{e}\langle j\rangle) \tag{3.34}$$

to obtain the physical components of the stress in the regions of nonequilibrium. We find that

$$
\begin{aligned}
\underset{-}{e}_{<1>} \cdot \underset{=}{T}\, \underset{-}{e}_{<1>} &= T_{11}, \\
\underset{-}{e}_{<1>} \cdot \underset{=}{T}\, \underset{-}{e}_{<2>} &= m\, T_{12} - n\, T_{13}, \\
\underset{-}{e}_{<1>} \cdot \underset{=}{T}\, \underset{-}{e}_{<3>} &= n\, T_{12} + m\, T_{13}, \\
\underset{-}{e}_{<2>} \cdot \underset{=}{T}\, \underset{-}{e}_{<2>} &= m^2\, T_{22} + n^2\, T_{33} - 2\, mn\, T_{23}, \\
\underset{-}{e}_{<2>} \cdot \underset{=}{T}\, \underset{-}{e}_{<3>} &= mn\, (T_{22}-T_{33}) + (m^2-n^2) T_{23},
\end{aligned}
\tag{3.35}
$$

and $$\underset{-}{e}_{<3>} \cdot \underset{=}{T}\, \underset{-}{e}_{<3>} = n^2 T_{22} + m^2 T_{33} + 2mn\, T_{23},$$

where T_{ij} are the stresses with respect to the basis $\underset{-}{b}_{<i>}$ for which $\underset{=}{M}$ has its proper form. Substituting the T_{ij} stress components which we obtained in the previous section into (3.33), we obtain

$$
\begin{aligned}
T_{<11>} &= (\beta_o - \beta\nu^2 + \alpha \text{ grad } \nu \cdot \text{grad } \nu + 2\alpha\nu\nabla^2\nu) - 2\alpha \left(\frac{\partial\nu}{\partial x^1}\right)^2, \\
T_{<12>} &= m(\mu k - 2\alpha \frac{\partial\nu}{\partial x^1} \frac{\partial\nu}{\partial x^2}) + 2n\alpha \left(\frac{\partial\nu}{\partial x^1} \frac{\partial\nu}{\partial x^3}\right), \\
T_{<13>} &= n(\mu k - 2\alpha \frac{\partial\nu}{\partial x^1} \frac{\partial\nu}{\partial x^2}) - 2m\alpha \left(\frac{\partial\nu}{\partial x^1} \frac{\partial\nu}{\partial x^3}\right),
\end{aligned}
\tag{3.36}
$$

$$T_{<22>} = (m^2+n^2)(\beta_o - \beta\nu^2 + \alpha \text{ grad } \nu \cdot \text{ grad } \nu + 2\alpha\nabla^2\nu)$$

$$- 2\alpha \left(m^2 \left(\frac{\partial \nu}{\partial x^2}\right)^2 + n^2 \left(\frac{\partial \nu}{\partial x^3}\right)^2\right) + 4mn\alpha \frac{\partial \nu}{\partial x^2}\frac{\partial \nu}{\partial x^3},$$

$$T_{<23>} = -2mn\alpha \left[\left(\frac{\partial \nu}{\partial x^2}\right)^2 - \left(\frac{\partial \nu}{\partial x^3}\right)^2\right] - 2(m^2-n^2)\frac{\partial \nu}{\partial x^2}\frac{\partial \nu}{\partial x^3},$$

and

$$T_{<33>} = (n^2+m^2)(\beta_o - \beta\nu^2 + \alpha \text{ grad } \nu \cdot \text{ grad } \nu + 2\alpha\nu\nabla^2\nu)$$

$$- 2n^2\alpha \left(\frac{\partial \nu}{\partial x^2}\right)^2 - 2m^2\alpha \left(\frac{\partial \nu}{\partial x^3}\right)^2 - 4mn\alpha \frac{\partial \nu}{\partial x^2}\frac{\partial \nu}{\partial x^3}. \qquad (3.37)$$

$T_{<ij>}$ are the physical stress components with respect to the basis $\underset{=}{e}_{<i>}$ with which we defined curvilineal motions. Here the $\frac{\partial}{\partial x^i}$ represent partial derivatives with respect to the coordinates for which $\underset{=}{M}$ has the required form. They are related to the $\frac{\partial}{\partial \hat{x}^i}$ with which we defined curvilineal flow by the relations $\frac{\partial}{\partial x^1} = \frac{\partial}{\partial \hat{x}^1}$,

$$\frac{\partial}{\partial x^2} = \frac{1}{m}\frac{\partial}{\partial \hat{x}^2} + \frac{1}{n}\frac{\partial}{\partial \hat{x}^3}, \quad \text{and} \quad \frac{\partial}{\partial x^3} = -\frac{1}{n}\frac{\partial}{\partial \hat{x}^2} + \frac{1}{m}\frac{\partial}{\partial \hat{x}^3}. \qquad (3.38)$$

Thus (3.34) becomes

$$T_{<11>} = (\beta_o - \beta\nu^2 + \alpha\ \hat{\text{grad}}\ \nu \cdot \hat{\text{grad}}\ \nu + 2\alpha\nu\hat{\nabla}^2\nu) - 2\alpha\ (\frac{\partial\nu}{\partial\hat{x}^1})^2,$$

$$T_{<12>} = m(\mu k - 2\alpha \frac{\partial\nu}{\partial\hat{x}^1} (\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3})) + 2n \frac{\partial\nu}{\partial\hat{x}^1} (\frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}),$$

$$T_{<13>} = n(\mu k - 2\alpha \frac{\partial\nu}{\partial\hat{x}^1} (\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3})) - 2m \frac{\partial\nu}{\partial\hat{x}^1} (-\frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}),$$

$$\begin{aligned} T_{<22>} = {} & (m^2+n^2)(\beta_o - \beta\nu^2 + \alpha\ \hat{\text{grad}}\ \nu \cdot \hat{\text{grad}}\ \nu + 2\alpha\nu\hat{\nabla}^2\nu) \\ & - 2\alpha(m^2 (\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3})^2 + n^2 (-\frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3})^2) \\ & + 4mn\alpha (\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3})(-\frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}), \end{aligned} \qquad (3.39)$$

$$\begin{aligned} T_{<23>} = {} & -2mn\alpha\ [(\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3})^2 - (-\frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3})^2] \\ & -2\ (m^2-n^2)(\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3})(-\frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}), \end{aligned}$$

and

$$\begin{aligned} T_{<33>} = {} & (n^2+m^2)(\beta_o - \beta\nu^2 + \alpha\ \hat{\text{grad}}\ \nu \cdot \hat{\text{grad}}\ \nu + 2\alpha\nu\hat{\nabla}^2\nu) \\ & - 2n^2\alpha\ (\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3})^2 - 2m^2\alpha\ (-\frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3})^2 \end{aligned}$$

$$- 4mn\alpha \left(\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3}\right)\left(- \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}\right).$$

$\hat{\mathrm{grad}}\ \nu$ and $\hat{\nabla}^2$ are the gradient and Laplacian operators written in terms of $\hat{x}^1$, $\hat{x}^2$, and $\hat{x}^3$; i.e.,

$$\hat{\mathrm{grad}}\ \nu \cdot \hat{\mathrm{grad}}\ \nu = \left(\frac{\partial\nu}{\partial\hat{x}^1}\right)^2 + \left(\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3}\right)^2 + \left(- \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}\right)^2, \text{ and}$$

$$\hat{\nabla}^2\nu = \frac{\partial^2\nu}{\partial\hat{x}^{12}} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2}\left(\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3}\right) + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3}\left(\frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^3}\right)$$
$$- \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2}\left(- \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}\right) + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}\left(- \frac{1}{n}\frac{\partial\nu}{\partial\hat{x}^2} + \frac{1}{m}\frac{\partial\nu}{\partial\hat{x}^3}\right).$$

Our representation for the form of the physical stress in the regions of nonequilibrium for curvilineal motions of Coulomb granular materials is complete.

CHAPTER IV

SPECIAL FLOW PROBLEMS

Flow Through a Channel

A motion is a steady shearing flow if the velocity field has the form

$$v\langle x\rangle = 0, \quad v\langle y\rangle = v(x), \quad v\langle z\rangle = 0 \tag{4.1}$$

in a Cartesian coordinate system x,y,z.

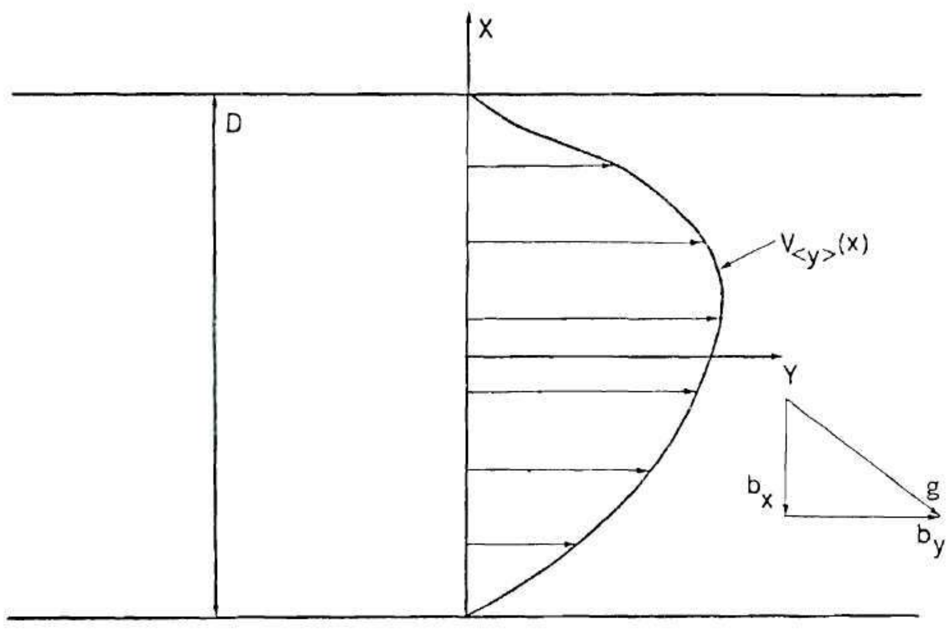

Figure 1. Channel Flow

These flows are curvilineal with

$$v^2(\hat{x}^1) = v(x), \quad v^1 = v^3 = 0, \tag{4.2}$$

and $e_1 = e_2 = e_3 = 1$. Then

$$k = \left(\frac{1}{e_1}\right) \sqrt{(v^2)_1\ e_2{}^2 + (v^3)_1\ e_3{}^2} = (v^2(x))_1 = \frac{dv(x)}{dx}. \tag{4.3}$$

For this problem we have the convenient simplification that $\underline{b}<i>$ and $\underline{e}<i>$ are equivalent. Additionally, we have $m = 1$ and $n = 0$.

It would be reasonable to assume that $\nu = \nu(x,y)$ only. In this case continuity requires that

$$\cancelto{0}{\frac{\partial \nu}{\partial t}} + \underline{v} \cdot \text{grad}\ \nu = 0. \tag{4.4}$$

However, since

$$\underline{v} = v(x)\underline{j}, \tag{4.5}$$

we have

$$v(x)\underline{j} \cdot \left(\frac{\partial \nu}{\partial x}\ \underline{i} + \frac{\partial \nu}{\partial y}\ \underline{j}\right) = v(x)\ \frac{\partial \nu}{\partial y}\ . \tag{4.6}$$

Also, since

$$v(x) \neq 0, \quad \frac{\partial \nu}{\partial y} = 0. \tag{4.7}$$

This implies that $\nu = \nu(x)$ only.

Therefore our stress tensor $\underline{\underline{T}} = \underline{\underline{T}}_E + \underline{\underline{T}}^o$ will be a function of x only.

From the results of the previous chapter, we obtain the following representation for the stress tensor in the regions of nonequilibrium:

$$\underline{\underline{T}} = \mu \begin{vmatrix} 0 & \frac{dv(x)}{dx} & 0 \\ \frac{dv(x)}{dx} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \left(\beta_o - \beta\nu^2 + \alpha\left(\frac{d\nu}{dx}\right)^2 + 2\alpha\nu\frac{d^2\nu}{dx^2}\right) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 2\alpha \begin{vmatrix} \left(\frac{d\nu}{dx}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}. \tag{4.8}$$

Recall that Cauchy's Equation for the balance of linear momentum is expressed in the form

$$\operatorname{div} \underline{\underline{T}} + \gamma\nu\underline{b} = \gamma\nu\underline{\ddot{x}} . \tag{4.9}$$

Let us consider only accelerationless motions. Then equating the right hand side of (4.9) to zero and inserting our representation for the stress gives

$$\frac{d}{dx}\,[\beta_o - \beta\nu^2 - \alpha(\frac{d\nu}{dx})^2 + 2\alpha\nu\,\frac{d^2\nu}{dx^2}] + \gamma\nu b_x = 0 \qquad (4.10)$$

and

$$\mu\,\frac{d^2v(x)}{dx^2} + \alpha\nu b_y = 0. \qquad (4.11)$$

Note that (4.10) contains only ν. Our equations in $v(x)$ and ν have become partially uncoupled. We can now solve directly for ν from (4.10). Differentiating, we obtain

$$-2\beta\nu\,\frac{d\nu}{dx} - 2\alpha\,\frac{d\nu}{dx}\,\frac{d^2\nu}{dx^2} + 2\alpha\,\frac{d\nu}{dx}\,\frac{d^2\nu}{dx^2} + 2\alpha\nu\,\frac{d^3\nu}{dx^3} + \gamma\nu b_x = 0. \qquad (4.12)$$

Note that a nonlinear term has cancelled out. Factoring out a ν and eliminating the case where $\nu \equiv 0$, we have

$$\frac{d^3\nu}{dx^3} - \frac{\beta}{\alpha}\,\frac{d\nu}{dx} + \frac{\gamma b_x}{2\alpha} = 0. \qquad (4.13)$$

The solution of this homogeneous equation with $b_x = 0$ is

$$\nu(x) = A\,\cosh\sqrt{\frac{\beta}{\alpha}}\,x + B\,\sinh\sqrt{\frac{\beta}{\alpha}}\,x + C. \qquad (4.14)$$

Nondimensionalizing x, we have $\bar{x} = \frac{2x}{D}$, D being the channel width. Then

$$\nu(x) = A \cosh L\bar{x} + B \sinh L\bar{x} + C, \tag{4.15}$$

where

$$L = \sqrt{\frac{\beta}{\alpha}}\left(\frac{D}{2}\right). \tag{4.16}$$

Now we can solve for the particular solution by letting

$$\nu_p = Ex. \tag{4.17}$$

Then
$$\frac{\beta}{\alpha} E = -\frac{\gamma b_x}{2\alpha} .$$

Now we define an angle ϕ as shown in Figure 2,

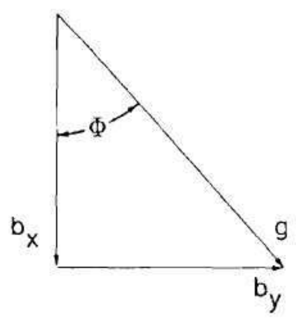

Figure 2. The Angle of Inclination

so that $b_x = g \cos\phi$ and $b_y = g \sin\phi$. Then

$$E = \frac{\gamma g}{\beta} \frac{\cos\phi}{2} . \tag{4.18}$$

Therefore

$$\nu_p = M \frac{\cos\phi}{2} \bar{x}, \tag{4.19}$$

where

$$M = \frac{\gamma g}{\beta} \left(\frac{D}{2}\right). \tag{4.20}$$

Therefore our complete solution for $\nu(\bar{x})$ is

$$\nu(\bar{x}) = A \cosh L\bar{x} + B \sinh L\bar{x} + C + M \frac{\cos\phi}{2} \bar{x}. \tag{4.21}$$

Now we can solve for the velocity from (4.11). Integrating twice gives

$$v(\bar{x}) = \frac{-\gamma g}{\mu} \Big[(\sin\phi)\Big(\frac{A}{L^2} \cosh L\bar{x} + \frac{B}{L^2} \sinh L\bar{x} + \frac{1}{2} C\bar{x}^2\Big) + D\bar{x} + E + \frac{M \cos\phi \sin\phi}{12} \bar{x}^3\Big]. \tag{4.22}$$

Nondimensionalizing the velocity then gives

$$\bar{v}(x) = \frac{-\gamma_g}{\mu}\left(\frac{D}{2}\right)^2 \left[(\sin\phi)\left(\frac{A}{L^2}\cosh L\bar{x} + \frac{B}{L^2}\sinh L\bar{x} + \frac{1}{2}C\bar{x}^2\right)\right.$$

$$\left. + D\bar{x} + E + M\frac{\cos\phi\sin\phi}{12}\bar{x}^3\right]. \qquad (4.23)$$

We have obtained general expressions for $\bar{v}(\bar{x})$ and $\nu(\bar{x})$ in the regions of nonequilibrium for the inclined channel gravity flow problems. The above results yield the same results as those obtained by Goodman and Cowin [1971, 1] for the special case of vertical flow. However, Goodman and Cowin assumed that a single plug of material in equilibrium would form in the center of the channel. It may be argued that other plug forms are possible, including a solution with regions of equilibrium occurring at the walls of the channel.* Examples of other possibilities are given in Figure 3.

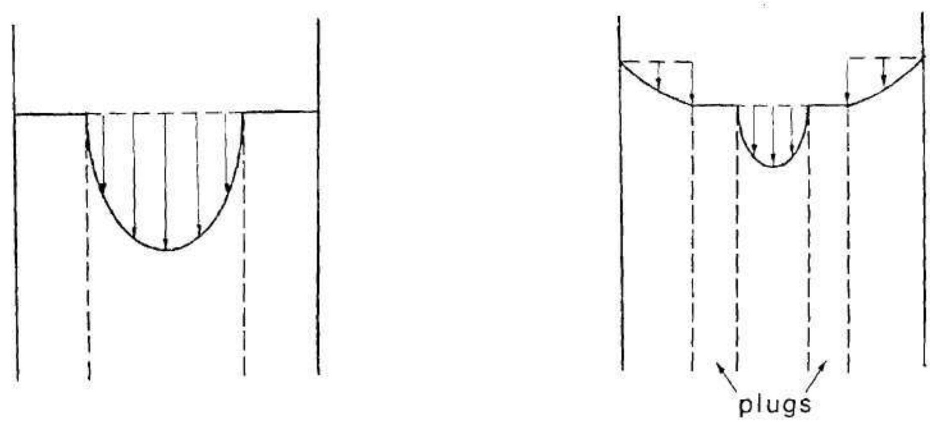

Figure 3. Possible Motions of Granular Materials

*See Appendix C for a discussion on the nonuniqueness of solutions.

Furthermore, there is no reason to assume that the distribution of plugs will necessarily be symmetric about the centerline for inclined channel flow.

Let us now consider the possible flow represented in Figure 3(a). We will leave the analysis in dimensional form to preserve clarity in the solution.

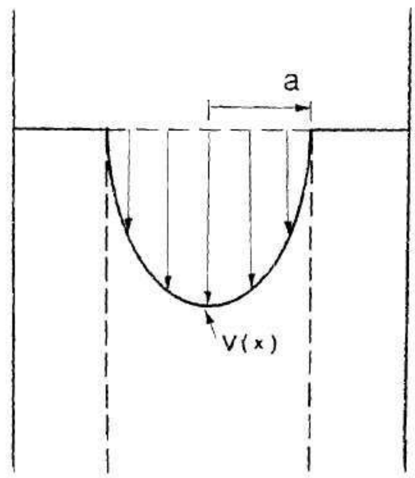

Figure 4. Channel Flow with Central Nonequilibrium Region

First we assume symmetry about the centerline. This requires that B = D = 0. Note that we assume adherence at the stationary channel walls. Since the velocity must then be zero throughout the regions of equilibrium we have

$$v(x)\Big|_{x = \pm a} = 0. \tag{4.24}$$

We also have the yield condition for the transition from regions of equilibrium to nonequilibrium, which requires

$$\frac{dv(x)}{dx}\Big|_{x = \pm a} = 0. \tag{4.25}$$

Let us also assume that we are given the value of

$$\nu(x)\Big|_{x = 0} = \nu(0). \tag{4.26}$$

Finally, let $L = \sqrt{\frac{\beta}{\alpha}}$ for the purposes of the solution.

The general form for the velocity field in the regions of nonequilibrium under the conditions of vertical flow and symmetric flow about the centerline is

$$v(x) = \frac{-\gamma_g}{\mu L^2} [A \cosh Lx + \frac{CL^2}{2} x^2 + E]. \tag{4.27}$$

Hence

$$\frac{dv(x)}{dx} = \frac{-\gamma_g}{\mu} [\frac{A}{L} \sinh Lx + Cx]. \tag{4.28}$$

Now imposing the condition

$$\left.\frac{dv(x)}{dx}\right|_{x\,=\,\pm a} = 0 = \frac{-\gamma g}{\mu}\left[\frac{A}{L}\sinh La + Ca\right] \tag{4.29}$$

yields the result that

$$C = \frac{-A}{aL}\sinh La. \tag{4.30}$$

Thus

$$v(x) = \frac{-\gamma g}{\mu}\left[\frac{A}{L^2}\cosh Lx \frac{-A}{2aL}\sinh La\, x^2 + E\right]. \tag{4.31}$$

Now we have

$$v(a) = 0 = A\left(\frac{1}{L^2}\cosh La - \frac{1}{2aL}(\sinh La)(a^2)\right) + E, \tag{4.32}$$

so

$$E = -A\left(\frac{1}{L^2}\cosh La - \frac{a}{2L}\sinh La\right), \tag{4.33}$$

and

$$v(x) = \frac{-\gamma g}{\mu} A\left[\frac{1}{L^2}(\cosh Lx + \cosh La) + \left(\frac{\sinh La}{2aL}\right)(a^2 - x^2)\right]. \tag{4.34}$$

Using (4.30) to reduce the general form of $v(x)$, we have

$$\nu(x) = A[\cosh Lx - \frac{\sinh La}{La}]. \tag{4.35}$$

Thus

$$\nu(x)\Big|_{x=o} = \nu(0) = A(\frac{La-\sinh La}{La}). \tag{4.36}$$

This yields us the solutions

$$\nu(x) = \nu(0)\ [\frac{La \cosh Lx - \sinh La}{La - \sinh La}], \quad -a < x < a \tag{4.37}$$

and

$$v(x) = \frac{-\gamma_g \nu(0) La}{\mu(La-\sinh La)} [\frac{1}{L^2} (\cosh Lx - \cosh La) + (\frac{\sinh La}{2aL})(a^2-x^2)],$$

$$-a < x < a. \tag{4.38}$$

In addition, we could specify that we are given the magnitude of the velocity at the centerline to obtain an equation to determine a* for given values of $\gamma/g, \nu(0)$, and L. However, if we investigate the result obtained in (4.37), we find that there is no need to proceed any further.

We notice an interesting result if we consider the

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*When the parameter a appears in a sentence, we underscore it to distinguish it from the words in the text. When it appears in an equation we write simply "a."

quantity La cosh Lx - sinh La, which is included in the bracketed term in (4.37). For $x = 0$, the term becomes La - sinh La, which must be less than zero because both L and a are assumed to be positive. For $x = a$, the term becomes La cosh La - sinh La, which is strictly positive. Thus the representation for ν always experiences a sign change over the interval $0 < x < a$, which is impossible because we have defined ν so that $0 \leq \nu(x) \leq 1$. Therefore flows of this type cannot exist.

Steady Helical Flows

A motion whose velocity field has the contravariant components

$$v^r = 0, \quad v^z = u(r), \quad v^\theta = w(r) \tag{4.39}$$

in a cylindrical coordinate system $\hat{x}^1 = r$, $\hat{x}^2 = z$, $\hat{x}^3 = \theta$ is a steady helical flow. This type of flow is curvilineal with

$$v^1(\hat{x}^1) = 0, \quad v^2(\hat{x}^1) = u(r), \quad v^3(\hat{x}^1) = w(r),$$

$$e_1 = e_r = 1, \quad e_2 = e_z = 1 \quad \text{and} \quad e_3 = e_\Theta = r. \tag{4.40}$$

For steady helical motions we have

$$k = \sqrt{(u^1)^2 + (rw^1)^2}\ , \tag{4.41}$$

$$m = \frac{u^1}{k} = \frac{u^1}{\sqrt{(u^1)^2 + (rw^1)^2}}\ , \tag{4.42}$$

$$n = \frac{w^1 r}{k} = \frac{w^1 r}{\sqrt{(u^1)^2 + (rw^1)^2}}\ , \tag{4.43}$$

and

$$\underline{\underline{M}} = \begin{vmatrix} 0 & 0 & 0 \\ u^1 & 0 & 0 \\ rw^1 & 0 & 0 \end{vmatrix} . \tag{4.44}$$

We can obtain some general results from the continuity equation before considering the problem directly. For steady helical motions, the continuity equation states that

$$\underline{v} \cdot \text{grad}\ \nu = 0 \tag{4.45}$$

which becomes

$$rv^\theta(r)\ \frac{\partial \nu}{r\partial\theta} + v^z(r)\ \frac{\partial \nu}{\partial z} = 0. \tag{4.46}$$

Suppose in general that ν has a dependence on all three position coordinates and that its solution may be expressed in a separable form, so that

$$\nu(r,\theta,z) = R(r)\Theta(\theta)Z(z). \tag{4.47}$$

If we put this into (4.46) and eliminate R, we have

$$v^{\theta}(r)\Theta'(\theta)Z(z) + V^{z}(r)\Theta(\theta)Z'(z) = 0. \tag{4.48}$$

For $v^z(r) \neq 0$, $\Theta(\theta) \neq 0$ and $Z(z) \neq 0$,

$$\frac{v^{\theta}(r)\Theta'(\theta)}{v^{z}(r)\Theta(\theta)} = \frac{-Z'(z)}{Z(z)} = \pm\lambda^2.^{*} \tag{4.49}$$

For solutions we have

$$Z'(z) \pm \lambda^2 Z(z) = 0, \tag{4.50}$$

or

$$Z(z) = ae^{\pm\lambda^2 z} \ . \tag{4.51}$$

Therefore in general, for $v^{\theta}(r) \neq 0$ and $v^z(r) \neq 0$, the z dependence of ν must be of an exponential form, assuming that

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*Later considerations dictate that the separation parameters λ^2 and $K_1{}^2$ must both be real.

$\nu = \nu(r,\theta,z)$. From (4.49), we must also have

$$\frac{v^{\theta}(r)}{v^{z}(r)} = \frac{\pm\lambda^{2}\Theta(\theta)}{\Theta'(\theta)} = \pm K_1^{2}, \qquad (4.52)$$

so

$$v^{\theta}(r) = \pm K_1^{2}\, v^{z}(r), \qquad (4.53)$$

and

$$\Theta(\theta) = Be(\pm \frac{\lambda^{2}}{K_1^{2}}\, \theta). \qquad (4.54)$$

Therefore the θ dependence of ν is likewise of an exponential form, and $v^{\theta}(r)$ and $v^{z}(r)$ must satisfy (4.53).

Now suppose $v^{z}(r) \equiv 0$, $v^{\theta}(r) \not\equiv 0$. Then returning to (4.46), we have $\frac{\partial\nu}{\partial\theta} = 0$. Therefore, if $v^{z}(r) \equiv 0$, then $\nu = \nu(z,r)$ only.

We may alternatively suppose that $v^{z}(r) \equiv 0$, $v^{\theta}(r) \not\equiv 0$. Then (4.46) implies that $\frac{\partial\nu}{\partial z} = 0$, so that $\nu = \nu(\theta,r)$ only.

We will assume that $\nu = \nu(r,z)$ only for the problems we attempt to solve. Then by (4.46), we have $v^{z}\,\frac{\partial\nu}{\partial z} = 0$. Therefore, $\frac{\partial\nu}{\partial z} = 0$, and the assumption that $\nu = \nu(r,z)$ reduces to the requirement that $\nu = \nu(r)$ only. Knowing that the velocity field and ν are dependent only on r, we may conclude from the constitutive equation for Coulomb granular materials that $\underline{\underline{T}} = \underline{\underline{T}}(r)$. The stress field in the regions of nonequilibrium will be given by

$$\underline{\underline{T}} = \mu \begin{vmatrix} 0 & u^1 & w^1 r \\ u^1 & 0 & 0 \\ w^1 r & 0 & 0 \end{vmatrix} + \left(\beta_o - \beta\nu^2 + \alpha\left(\frac{d\nu}{dr}\right)^2 + 2\alpha\nu \frac{d}{dr}\left(\frac{1}{r}\frac{d\nu}{dr}\right)\right) \underline{\underline{I}} - 2\alpha \begin{vmatrix} \left(\frac{d\nu}{dr}\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} . \qquad (4.55)$$

We can put this stress relation into the dynamical equation to obtain the "modified" Navier-Stokes equation for steady helical incompressible motions of granular materials. Recall from (1.25) that the general form of the "modified" Navier-Stokes equation is

$$\gamma\nu\dot{\underline{v}} = \gamma\nu\underline{b} - 2\beta\nu \text{ grad } \nu + 2\alpha\nu \text{ grad}(\nabla^2\nu) + \mu\nabla^2\underline{v}. \qquad (4.56)$$

If we specialize this to the case of steady helical motions, we have, writing $v_\theta = w^1 r$ to simplify the subsequent analysis,

$$-\gamma\nu \frac{v_\theta^2}{r} = \gamma\nu b_r - 2\beta\nu \frac{d\nu}{dr} + 2\alpha\nu \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\left(r \frac{d\nu}{dr}\right)\right),$$

$$0 = \gamma\nu b_z + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r \frac{d(u^1)}{dr}\right)\right], \text{ and} \qquad (4.57a,b,c)$$

$$0 = \gamma\nu b_\theta + \mu\left[\frac{1}{r}\frac{d}{dr}\left(r \frac{dv_\theta}{dr}\right) - \frac{v_\theta}{r^2}\right].$$

Special Cases of Steady Helical Motions

I. Motion of granular materials between rotating concentric cylinders--exact and inexact solutions

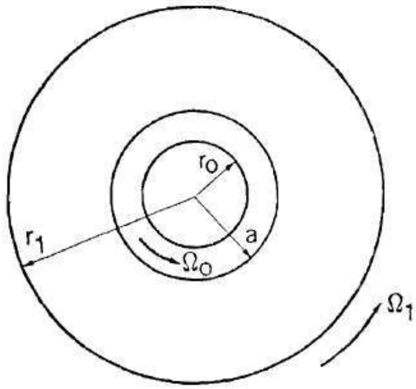

Figure 5. Flow Between Rotating Concentric Cylinders

We will first investigate the problem of the circular motion of a granular material between two horizontal concentric cylinders. For motions of this type we have the condition that $v^z \equiv 0$. Hence equations (4.57 a,c) with the assumption that $b_r = b_\theta = 0$ are the two equations pertinent to this problem. Equations (4.57 a,c) are partly uncoupled, with only v_θ appearing in (4.57c) and both variables occurring in (4.57a). We can solve for v_θ from (4.57c) and obtain the solution of the homogeneous equation in ν from (4.57a) by setting $v_\theta = 0$. The particular solution for ν can then be obtained by inserting our derived velocity expression into the right hand side of (4.57a), assuming an appropriate form for ν and then solving for the coefficients.

We know from the solution to this problem for a Navier-Stokes fluid that to obtain the velocity from (4.57c), we can assume a solution of the form

$$v_\theta = Ar^\alpha. \tag{4.58}$$

We obtain

$$v_\theta = Ar + \frac{B}{r} . \tag{4.59}$$

Now we find the solution of the homogeneous equation in ν. Setting $v_\theta = 0$ in (4.57a) and assuming $b_r = 0$, we have

$$0 = \nu[\frac{d}{dr} (\frac{1}{r} \frac{d\nu}{dr} + \frac{d^2\nu}{dr^2}) - \frac{\beta}{\alpha} \frac{d\nu}{dr}]. \tag{4.60}$$

Eliminating the case $\nu \equiv 0$ and integrating once with respect to r, we have

$$K_2 = \frac{d^2\nu}{dr^2} + \frac{1}{r} \frac{d\nu}{dr} - \frac{\beta}{\alpha} \nu. \tag{4.61}$$

The homogeneous part of (4.61) is the modified Bessel's equation of order 0. Thus we obtain a solution of the form

$$\nu_H(r) = C_1 I_o\left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_2 K_o\left(\sqrt{\frac{\beta}{\alpha}}\, r\right). \tag{4.62}$$

Now assume $\nu_p(r) = D$ and find from (4.61) that

$$D = -\frac{\alpha}{\beta} K_2 = C_3. \tag{4.63}$$

Therefore our solution of the homogeneous equation in ν is

$$\nu = C_1 I_o\left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_2 K_o\left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_3. \tag{4.64}$$

To find the particular solution for ν from equation (4.57a), we put in our derived expression for V_θ. We have

$$-2\beta \frac{d\nu}{dr} + 2\alpha \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r \frac{d\nu}{dr}\right)\right] = -\gamma\left[A^2 r + \frac{2AB}{r} + \frac{B^2}{r^3}\right] \tag{4.65}$$

Again we integrate once with respect to r:

$$-2\beta\nu + 2\alpha\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\nu}{dr}\right)\right) = -\gamma\left[\frac{A^2 r^2}{2} + 2AB\ell n\, r - \frac{B^2}{2r^2} + C\right]. \tag{4.66}$$

To find the particular solution of (4.66), assume that ν_p has the form

$$\nu_p = D + Er + Fr^2 + G\ell n\, r. \tag{4.67}$$

Putting this into the left hand side of (4.66), we get

$$-2\beta\ (D + Er + Fr^2 + G\ln r) + 2\alpha\ (\frac{E}{r} + 4F)$$

$$= -\gamma\ [\frac{Ar^2}{2} + 2AB\ln r - \frac{B^2}{2r^2} + C]. \qquad (4.68)$$

Note that the $\frac{1}{r^2}$ terms on the right hand side have cancelled. This means that no particular solution for ν may be assumed so that a nonzero $\frac{1}{r^2}$ term will appear on the left hand side of (4.68). Therefore we can find no separable exact solutions to the equations for which $B \neq 0$. The only separable exact solutions possible are those for which $B = 0$. This means that the only separable exact solutions which can be obtained for motions of granular materials between rotating concentric cylinders will be rigid body rotations, since $v_\theta = Ar$ if $B = 0$.

Let us obtain the exact solution, assuming $B = 0$. If we equate r^2 terms on both sides of (4.68), we obtain

$$F = \frac{\gamma A^2}{4\beta}\ . \qquad (4.69)$$

If we equate r terms, we have

$$E = 0. \qquad (4.70)$$

When we equate constant terms we get

$$D = \frac{\gamma c}{2\beta} + \frac{\alpha\gamma A^2}{\beta^2} . \tag{4.71}$$

Finally, if we equate $\ell n\ r$ terms, we find that

$$I = 0. \tag{4.72}$$

Therefore the only separable exact solutions for regions of nonequilibrium for this type of flow are

$$v_\theta = Ar, \tag{4.73}$$

and

$$\nu = C_1 I_o \left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_2 K_o \left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_4 + \frac{\alpha\gamma A^2}{\beta^2} + \frac{\gamma A^2 r^2}{4\beta}. \tag{4.74}$$

We have put $C_4 = C_3 + \frac{\gamma C}{2\beta}$ since both C_3 and C are arbitrary constants of integration.

The yield condition we will employ for this type of flow is

$$\left.\frac{dv_\theta}{dr}\right|_{r=a} = 0. \tag{4.75}$$

However, notice that this implies the entire region must be in equilibrium if any part is in equilibrium, since $\frac{dv_\theta}{dr} = 0$ implies that A = 0. Hence we can solve rigid body motions with an exact separable solution only if the entire region of granular material is in nonequilibrium.

For inexact solutions to the classical Navier-Stokes equations, the inertia term $\frac{\gamma \nu v_\theta^2}{r}$ is assumed to be negligible. Let us make the same assumption here so that we can formulate inexact solutions to the equations for motions other than rigid body rotations.

If we assume that the inertia term is negligible, then our solutions for v_θ and ν become

$$v_\theta = Ar + \frac{B}{r} \tag{4.76}$$

and

$$\nu = C_1 I_o \left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_2 K_o \left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_4. \tag{4.77}$$

Notice that the solutions are completely uncoupled, so that varying the coefficients of one of the solutions will not affect the other solution.

We will next solve a problem for which we can obtain both an exact and an inexact solution so that we can compare the results for one particular case.

Consider the case where there is no inner cylinder, and the outer cylinder which forms the boundary of the granular material is rotating with angular velocity Ω_1 as in Figure 6.

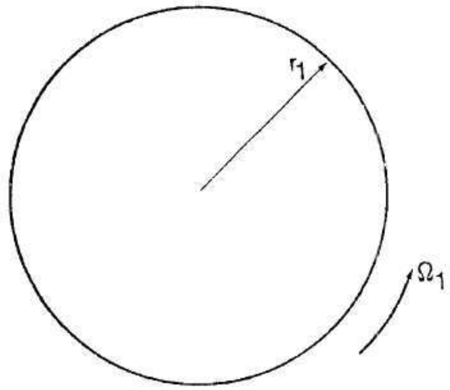

Figure 6. Flow in a Rotating Cylinder

The velocity will simply be $v_\theta = \Omega_1 r$, which is an exact solution. Assume that we are given

$$\nu(r)\Big|_{r=0} = \nu(0) \quad \text{and} \quad \nu(r)\Big|_{r=r_1} = \nu(r_1). \tag{4.78}$$

Since ν must be finite at the origin, $C_2 = 0$, because $K_o(z)$ becomes unbounded as $z \to 0$. Also, $I_o(0) = 1$. Then

$$\nu(0) = C_1 + C_4 + \frac{\alpha\gamma}{\beta^2}\,\Omega_1{}^2. \tag{4.79}$$

So

$$C_4 = \nu(0) - C_1 - \frac{\gamma\alpha\Omega_1{}^2}{\beta^2},$$

and

$$\nu(r) = C_1(I_o(\sqrt{\tfrac{\beta}{\alpha}}\, r)-1) + \nu(0) + \frac{\gamma\Omega_1{}^2}{4\beta} r^2. \qquad (4.80)$$

Now

$$\nu(r_1) = C_1(I_o(\sqrt{\tfrac{\beta}{\alpha}}\, r_1)-1) + \nu(0) + \frac{\gamma\Omega_1{}^2}{4\beta} r_1{}^2. \qquad (4.81)$$

If we solve for C_1 and write out our final solution for ν and v_θ, we have

$$\underset{o<r<r_1}{v_\theta} = \Omega_1 r, \qquad (4.82)$$

and

$$\underset{o<r<r_1}{\nu(r)} = [\nu(r_1)-\nu(0) - \frac{\gamma\Omega_1{}^2 r_1{}^2}{4\beta}][\frac{I_o(\sqrt{\frac{\beta}{\alpha}}\, r)-1}{I_o(\sqrt{\frac{\beta}{\alpha}}\, r_1)-1}] + \nu(0) + \frac{\gamma\Omega_1{}^2 r^2}{4\beta}.$$

This is an exact solution. Now let us see what results we obtain from the inexact solution. We have

$$v_\theta = Ar + \frac{B}{r}, \qquad (4.83)$$

with the conditions that $v_\theta(0) = 0$ and $v_\theta(r_1) = \Omega_1 r_1$. Thus we obtain $v_\theta(r) = \Omega_1 r$ as our velocity solution, which is identical to the velocity expression for the exact solution. Now we have

$$\nu(r) = C_1 I_o \left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_2 K_o \left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C_4 \tag{4.84}$$

as our inexact solution for ν, under the conditions that

$$\nu(r)\Big|_{r=o} = \nu(0) \quad \text{and} \quad \nu(r)\Big|_{r=r_1} = \nu(r_1). \tag{4.85}$$

Again $C_2 = 0$ because $\lim\limits_{r \to 0} \nu(r)$ must be bounded. We impose our first condition, so that

$$\nu(0) = C_1 + C_4 \tag{4.86}$$

Therefore

$$\nu(r) = C_1 \left(I_o \left(\sqrt{\frac{\beta}{\alpha}}\, r\right) - 1\right) + \nu(0). \tag{4.87}$$

Now

$$\nu(r_1) = C_1 \left(I_o \left(\sqrt{\frac{\beta}{\alpha}}\, r_1\right) - 1\right) + \nu(0), \tag{4.88}$$

$$0 < r < r_1 \quad .$$

Thus we find that

$$\nu(r)_{o<r<r_1} = [\nu(r_1)-\nu(0)]\left[\frac{I_o(\sqrt{\frac{\beta}{\alpha}}\, r)-1}{I_o(\sqrt{\frac{\beta}{\alpha}}\, r_1)-1}\right] + \nu(0) \tag{4.89}$$

is our solution for the inexact case.

Comparing (4.82) and (4.89), we note that the constant multiplying the

$$\frac{I_o(\sqrt{\frac{\beta}{\alpha}}\, r)-1}{I_o(\sqrt{\frac{\beta}{\alpha}}\, r_1)-1}$$

term is smaller in the exact theory by an amount equal to $\frac{\gamma\Omega_1{}^2 r_1{}^2}{4\beta}$, and that the exact solution has a dependence proportional to r^2 not present in the inexact solution. The key difference in the two solutions is that the exact solution for ν is dependent on the square of the magnitude of Ω_1, whereas the inexact solution yields the same function for ν independent of the angular velocity. We expect that the inexact solution will be close to the exact solution only for small angular velocities. For such cases, the terms depending on the angular velocity which are present in the exact solution are negligible. In Figure 7, we have plotted ν for the following set of constants: $\nu(0) = .25$, $\nu(1) = .75$, $\frac{\beta}{\alpha} = 1$, $\frac{\gamma}{\beta} = 2\ \frac{\text{sec}^2}{\text{ft}^2}$, $r_1 = 1'$, $\nu_1 = \nu$ inexact and

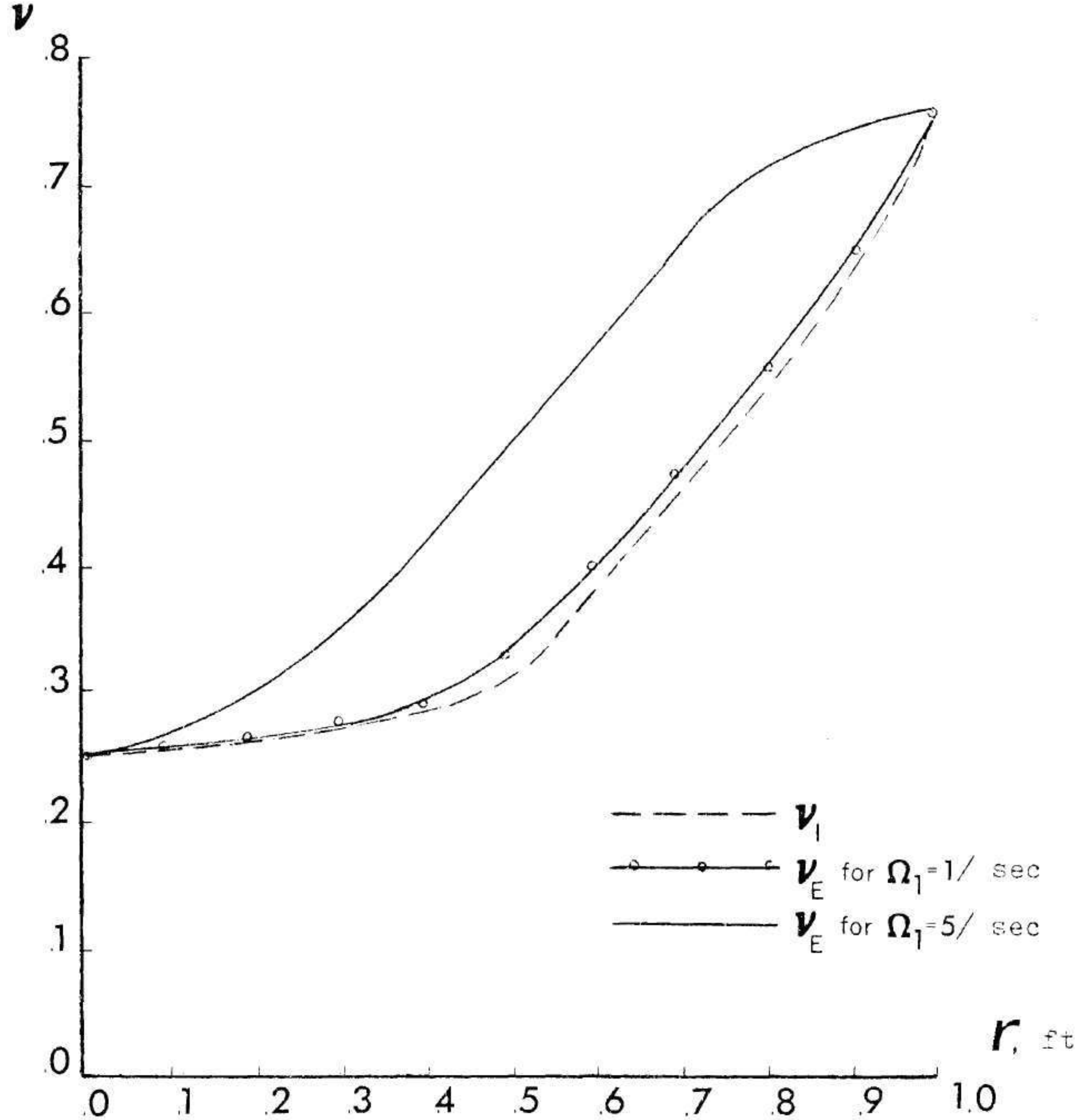

Figure 7. Exact vs. Inexact Solution For Cylinder Flow

$\nu_E = \nu_{exact}$. The inexact solution (which is independent of Ω_1) is plotted, and the exact solution is plotted for two different angular velocities. From this graph we see that the inexact solution is indeed close to the exact solution for $\Omega_1 = 1 \frac{rad}{sec}$, but that for $\Omega_1 = 5 \frac{rad}{sec}$ the results are no longer similar. We will use the fact that the results are very similar for a small Ω_1 in this case as justification for our consideration of another problem using the inexact solution where the magnitudes of the angular velocities of the cylinders are small.

Bearing in mind this restriction on the validity of our solution, we consider the problem of flow between concentric cylinders, as shown in Figure 8.

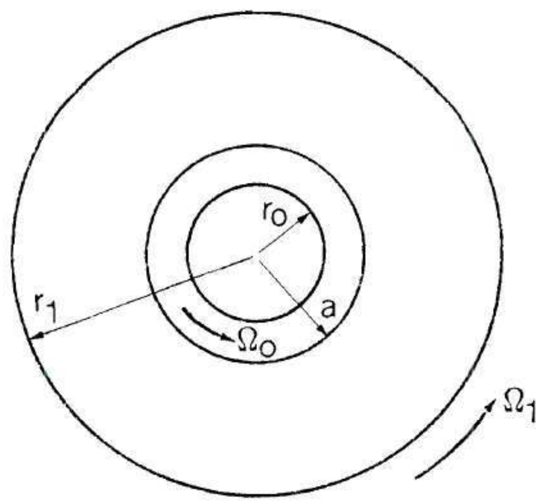

Figure 8. Flow Between Slowly Rotating Concentric Cylinders

We have a region of equilibrium surrounding the inner cylinder out to a radius $\underline{a}$. The region of nonequilibrium is defined to be over the interval $a < r < r_1$. The boundary conditions on the velocity are

$$v_\theta(a) = \Omega_o r_o, \quad v_\theta(r_1) = \Omega_1 r_1, \quad \text{and} \quad \left.\frac{dv_\theta(r)}{dr}\right|_{r=a} = 0. \quad (4.90)$$

The boundary conditions on ν are

$$\nu(r)\Big|_{r=a} = \nu(a) \quad \text{and} \quad \nu(r)\Big|_{r=r_1} = \nu(r_1). \qquad (4.91)$$

Let $L = \sqrt{\frac{\beta}{\alpha}}\, r_1$.

The fact that we use the value of ν at $r = a$ as a boundary condition on ν means that $\underline{a}$ must first be determined from the velocity and yield conditions before measurements determining $\nu(a)$ can be taken. Since the solutions for v_θ and ν are completely uncoupled in the inexact solution, this requirement poses no special problems.

We first solve for v_θ. The general solution must be of the form

$$v_\theta = Ar + \frac{B}{r}\ , \qquad (4.92)$$

so

$$v_\theta(r)\Big|_{r=r_1} = \Omega_1 r_1 = Ar_1 + \frac{B}{r_1} . \qquad (4.93)$$

Thus

$$B = (\Omega_1 - A)r_1^{\,2}, \qquad (4.94)$$

which gives us

$$v_\theta(r) = Ar + \frac{(\Omega_1 - A)r_1^{\,2}}{r} . \qquad (4.95)$$

Now

$$v_\theta(r)\Big|_{r=a} = \Omega_o r_o = Aa + \frac{(\Omega_1 - A)r_1^{\,2}}{a} , \qquad (4.96)$$

so that

$$A = \frac{\Omega_o r_o a - \Omega_1 r_1^{\,2}}{a^2 - r_1^{\,2}} . \qquad (4.97)$$

Thus the solution for $v_\theta(r)$ is

$$v_\theta(r) = [\frac{\Omega_o r_o a - \Omega_1 r_1^{\,2}}{a^2 - r_1^{\,2}}]r + [\frac{\Omega_1 a^2 - \Omega_o r_o a}{a^2 - r_1^{\,2}}] \frac{r_1^{\,2}}{r} . \qquad (4.98)$$

We must now apply the yield condition to determine $\underline{a}$. We have

$$\left.\frac{dv_\theta(r)}{dr}\right|_{r=a} = 0 = \Omega_o r_o a - \Omega_1 r_1{}^2 - (\Omega_1 a^2 - \Omega_o r_o a)\ \frac{r_1{}^2}{a^2}\ . \tag{4.99}$$

We can reduce this to a quadratic equation in a by an algebraic manipulation and by factoring out an a. We get

$$a_{1,2} = \frac{r_1\Omega_1}{r_o\Omega_o}\ (r_1 \pm \frac{1}{\Omega_1}\sqrt{\Omega_1{}^2 r_1{}^2 - \Omega_o{}^2 r_o{}^2}). \tag{4.100}$$

Then $\Omega_1{}^2 r_1{}^2 > \Omega_o{}^2 r_o{}^2$ must be a restriction on the types of permissible solutions, because the radical must be real. This condition may be rewritten as

$$\frac{\Omega_1{}^2}{\Omega_o{}^2}\ \frac{r_1{}^2}{r_o{}^2} > 1. \tag{4.101}$$

Thus the solution for a is of the form

$$a_{1,2} = N(r_1 \pm \varepsilon), \tag{4.102}$$

where $N > 1$ and $\varepsilon > 0$. This means that the choice $a_1 = N(r_1+\varepsilon)$ is physically unrealistic, since it yields a value for a that is greater than r_1, the radius of the outer cylinder. So we must have $a_2 = N(r_1-\varepsilon)$, and a_2 must be greater than r_o, the radius of the inner cylinder. This condition can be written as

$$\left(\frac{r_1}{r_o}\right)\left(\frac{\Omega_1}{\Omega_o}\right)\left(r_1 - \frac{1}{\Omega_1}\sqrt{\Omega_1{}^2 r_1{}^2 - \Omega_o{}^2 r_o{}^2}\right) > r_o. \tag{4.103}$$

We can rewrite this as

$$r_1 - \frac{1}{\Omega_1}\sqrt{\Omega_1{}^2 r_1{}^2 - \Omega_o{}^2 r_o{}^2} > \frac{r_o{}^2 \Omega_o}{r_1 \Omega_1} \tag{4.104}$$

or

$$\left(\frac{r_1{}^2\Omega_1 - r_o{}^2\Omega_o}{r_1}\right)^2 > \Omega_1{}^2 r_1{}^2 - \Omega_o{}^2 r_o{}^2. \tag{4.105}$$

Squaring out the term on the left hand side of (4.105) and simplifying gives

$$\left(\frac{r_o{}^2}{r_1{}^2} + 1\right)\Omega_o > 2\Omega_1 \tag{4.106}$$

as our second restriction on the solution. Therefore we have found that

$$a = \frac{r_1 \Omega_1}{r_o \Omega_o}\left(r_1 - \frac{1}{\Omega_1}\sqrt{\Omega_1{}^2 r_1{}^2 - \Omega_o{}^2 r_o{}^2}\right) \tag{4.107}$$

is the solution for a under the two restrictions that

$$\frac{\Omega_1{}^2 r_1{}^2}{\Omega_o{}^2 r_o{}^2} > 1 \quad \text{and} \quad \left(\frac{r_o{}^2}{r_1{}^2} + 1\right)\Omega_o > 2\Omega_1. \tag{4.108}$$

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which yields

$$\nu(r) = \left[\frac{K_o(L)\nu(a) - K_o(L\frac{a}{r_1})\nu(r_1)}{K_o(L)I_o(L\frac{a}{r_1}) - K_o(L\frac{a}{r_1})I_o(L)}\right]\left[\frac{I_o(L\frac{r}{r_1})K_o(L) - I_o(L)K_o(L\frac{r}{r_1})}{K_o(L)}\right]$$

$$+ \nu(r_1)\frac{K_o(L\frac{r}{r_1})}{K_o(L)} \qquad (4.114)$$

as the solution for ν to within an arbitrary constant. Consider a boundary value problem of this type, for which we take

$$r_o = 1.8 \text{ ft}, \qquad r_1 = 2.0 \text{ ft},$$

$$\Omega_o = 1\ \frac{\text{rad}}{\text{sec}}, \qquad \Omega_1 = .904\ \frac{\text{rad}}{\text{sec}} \qquad (4.115)$$

Let us see if this satisfies the restrictions. We have

$$\frac{\Omega_1}{\Omega_o}\frac{r_1}{r_o} = \left(\frac{.904}{1}\right)\left(\frac{2.0}{1.8}\right) = 1.004 > 1, \qquad (4.116)$$

so the first restriction is satisfied. Also,

$$\left(\frac{r_o^{\,2}}{r_1^{\,2}} + 1\right)\Omega_o = \left[\left(\frac{1.8}{2.0}\right)^2 + 1\right](1) = 1.81 \qquad (4.117)$$

and $2\Omega_1 = 2(.904) = 1.808 < 1.81$, so the second restriction is satisfied as well. If we solve for $\underline{a}$, we find that

$$a = 1.82 \text{ ft} . \tag{4.118}$$

The functional representations for v_θ and ν for this specific boundary value problem turn out to be

$$v_\theta(r) = (.4945\, r + \frac{1.638}{r}) \frac{\text{ft}}{\text{sec}},\ 1.82 < r < 2; \tag{4.119}$$

and

$$\nu(r) = 1.588\ [I_o\ (\tfrac{r}{2}) - 3.007\ K_o\ (\tfrac{r}{2})] + 1.78\ K_o\ (\tfrac{r}{2}),$$

$$1.82 < r < 2;$$

where we use the values $\nu(a) = .5$, $\nu(r_1) = .75$, and $L = 1$, and r is given in feet. We have plotted the velocity profile and volume distribution profile for this problem in Figure 9. The velocity varies within the narrow range $1.8 < v_\theta(r) < 1.808 \frac{\text{ft}}{\text{sec}}$. The curve representing the volume distribution function is nearly linear in this case throughout the region of nonequilibrium.

II. Longitudinal motion of granular materials between concentric vertical cylinders

For this type of motion, as illustrated in Figure 10, we have

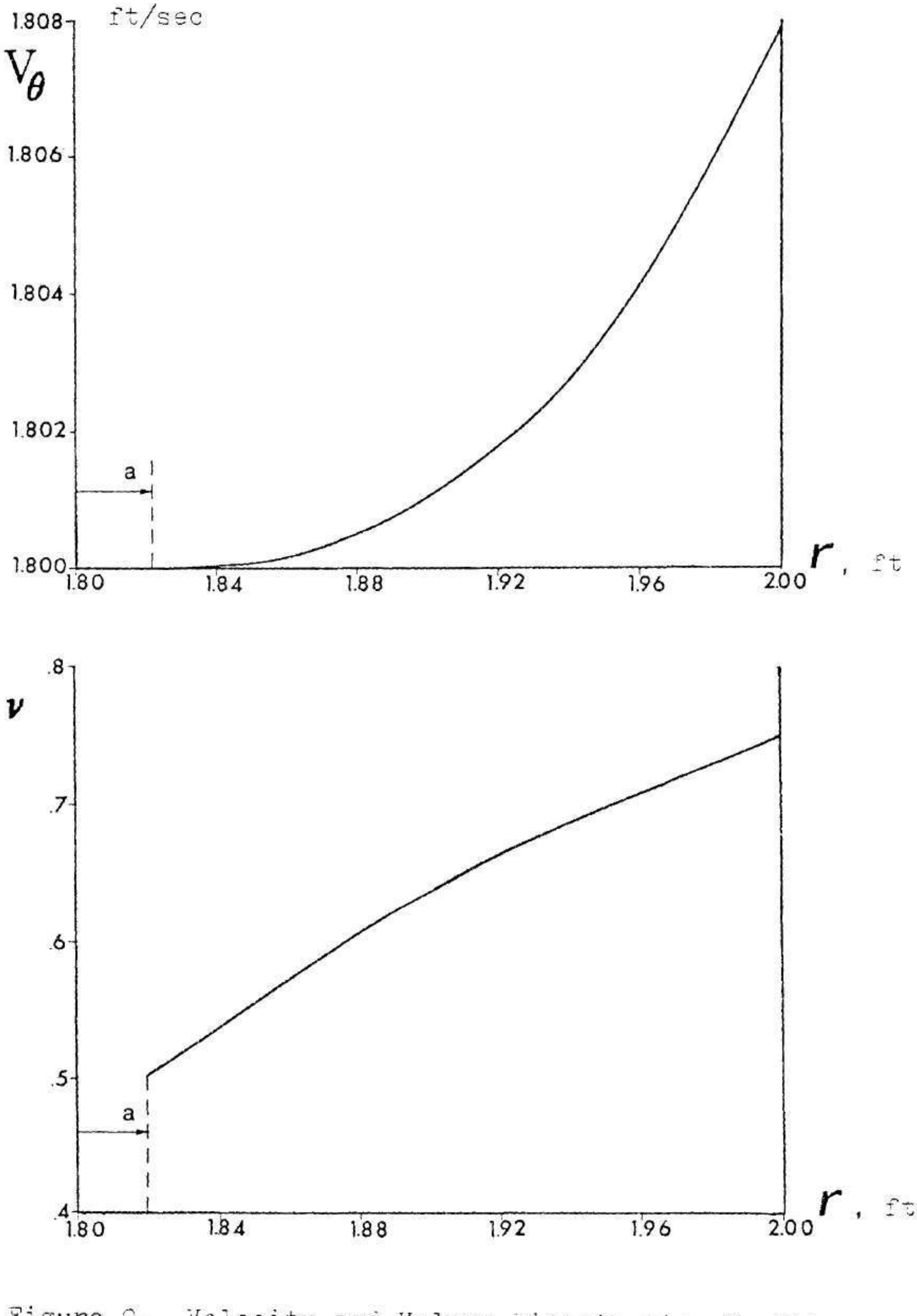

Figure 9. Velocity and Volume Distribution Profiles

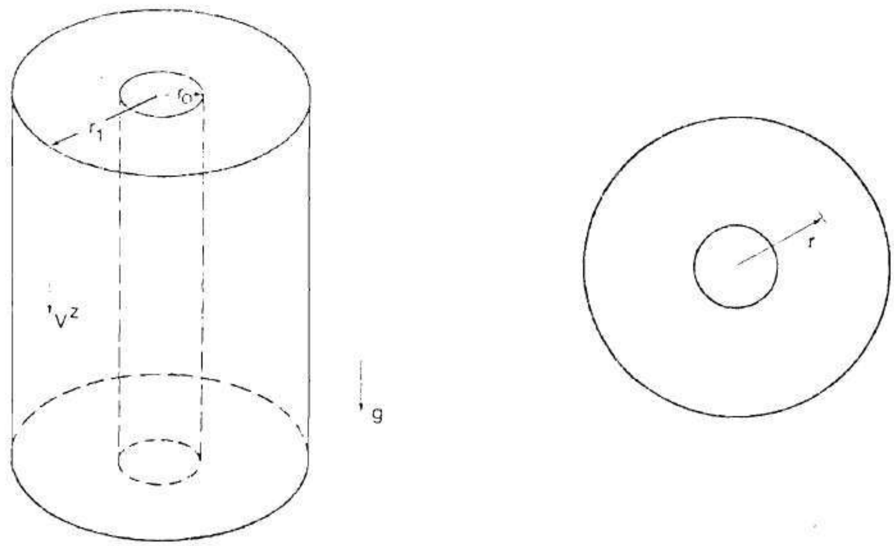

Figure 10. Longitudinal Motion Between Concentric Vertical Cylinders

$$v^r = 0, \quad v^z = v^z(r), \quad \text{and } v^\theta = 0; \quad \text{also} \quad \nu = \nu(r). \quad (4.120)$$

We assume that the cylinders are vertical, so that $b_z = g$. The equations which govern this type of motion are (4.57a,b). Using our restrictions on the possible motions given by (4.120), these equations become

$$\nu\{2\alpha[\frac{d}{dr}(\frac{1}{r}\frac{d}{dr}(r\frac{d\nu}{dr}))] - 2\beta\frac{d\nu}{dr}\} = 0$$

and (4.121a,b)

after simplification, where $v^z = u'(r)$.

$$\mu[\frac{1}{r}\frac{d}{dr}(r\frac{dv^z}{dr})] + \gamma\nu g = 0$$

Again, the equations become partly uncoupled. We can solve for ν from (4.121 a). We obtain the homogeneous equation in V^z from (4.121 b) by putting $\nu = 0$. Part of our solution for V^z is determined by solving this homogeneous equation. Finally, if we insert our expression for ν into (4.121 b), we get the particular solution for ν by assuming an appropriate form.

First we discard the case $\nu \equiv 0$ and integrate (4.121 a) with respect to r, which gives us

$$C_1 = -\frac{\beta}{\alpha}\nu + \frac{1}{r}\left[\frac{d\nu}{dr} + r\frac{d^2\nu}{dr^2}\right]. \tag{4.122}$$

The homogeneous equation obtained by setting $C_1 = 0$ is the modified Bessel's equation of order 0. Our particular solution will be an arbitrary constant. So

$$\nu(r) = A\, I_o\left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + B\, K_o\left(\sqrt{\frac{\beta}{\alpha}}\, r\right) + C. \tag{4.123}$$

We obtain the solution to the homogeneous equation determined from (4.121 b) by setting $\nu = 0$. Then we have

$$\frac{1}{r}\frac{dv^z}{dr} + \frac{d^2v^z}{dr^2} = 0. \tag{4.124}$$

If we let $w = \frac{dv^z}{dr}$, then (4.124) becomes

$$\frac{dw}{dr} + \frac{w}{r} = 0. \tag{4.125}$$

The solution is

$$w = \frac{D}{r}\,. \tag{4.126}$$

Then

$$v_H^z = D\ell nr + E. \tag{4.127}$$

To find the particular solution for v^z, we put our derived expression for ν into (4.121b) and rewrite to obtain

$$\frac{-\gamma_g}{\mu}\,(AI_o(Lr) + BK_o(Lr) + C) = \frac{d^2v^z}{dr^2} + \frac{1}{r}\,\frac{dv^z}{dr}\,, \tag{4.128}$$

where $L = \sqrt{\frac{\beta}{\alpha}}$. We try $v^z = FI_o(Lr) + GK_o(Lr) + Hr^2$ as the particular solution. Note that we have

$$\frac{d}{dr}\,(r\,\frac{dv^z}{dr}) = \frac{dv^z}{dr} + r\,\frac{d^2v^z}{dr^2} = rL^2(FI_o(Lr) + GK_o(Lr)) + 4\,Hr. \tag{4.129}$$

We therefore obtain

$$\frac{1}{r}\,\frac{d}{dr}\,(r\,\frac{dv^z}{dr}) = FL^2I_o(Lr) + GL^2K_o(Lr) + 4H. \tag{4.130}$$

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This is an exact solution.

Let us investigate possible Poiseuille flows, which are longitudinal motions down a single vertical pipe which we assume to have a constant circular cross section as in Figure 11.

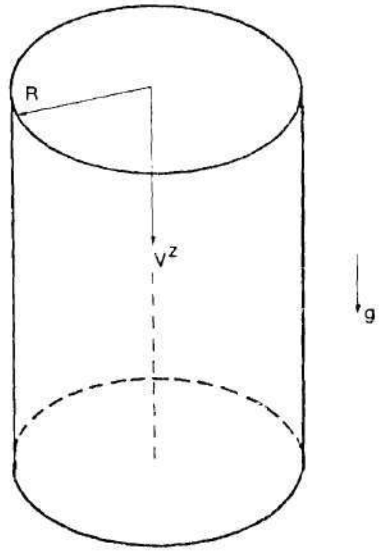

Figure 11. Poiseuille Flow

Recall that for the case of vertical channel flow we showed that there are no symmetric solutions with a central region of nonequilibrium. Let us see if we can obtain the same result in the case of Poiseuille flow. Figure 12 indicates the form of the symmetric solution for Poiseuille flow.

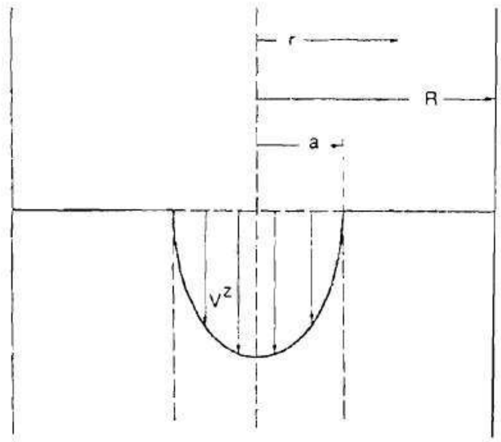

Figure 12. Poiseuille Flow with Central Nonequilibrium Region

We can use the general results for ν and v^z obtained in (4.123) and (4.135). For this problem, since ν and v^z must be bounded at $r = 0$, we must have $B = D = 0$ because $\ell n\, r$ and $K_0(Lr)$ become unbounded as r approaches zero. For our other boundary conditions, we have $\left.\frac{dv^z(r)}{dr}\right|_{r=a} = 0$, $v^z(a) = 0$, $\left.\nu(r)\right|_{r=a} = \nu(a)$, and $\left.v^z(r)\right|_{r=0} = v^z(0)$. The last condition allows us to determine the value of $\underline{a}$ for a given centerline velocity. Setting $B = 0$ in (4.135) and taking the derivative with respect to r gives

$$\left.\frac{dv^z(r)}{dr}\right|_{r=a} = \frac{-\gamma_g}{\mu L^2}\left(ALI_1(Lr) + \frac{2CrL^2}{4}\right)\Bigg|_{r=a} = 0, \qquad (4.136)$$

so

$$\frac{CL^2}{4} = \frac{-ALI_1(La)}{2a} . \qquad (4.137)$$

Then our velocity function becomes

$$v^z(r) = E - \frac{\gamma_g}{\mu L^2} A(I_o(Lr) - \frac{LI_1(La)}{2a} r^2). \qquad (4.138)$$

Now imposing the condition that $v^z(a) = 0$, we get

$$0 = \frac{-\gamma_g A}{\mu L^2} (I_o(La) - \frac{LI_1(La)a^2}{2a}) + E, \qquad (4.139)$$

which determines the constant E.

Using (4.137) to reduce our expression for ν and setting $B = 0$ gives

$$\nu(r) = A\{I_o(Lr) - \frac{4}{L^2} (\frac{LI_1(La)}{2a})\}. \qquad (4.140)$$

Now

$$\nu(r)\Big|_{r=a} = \nu(a) = A\{I_o(La) - \frac{2}{aL} (I_1(La))\}, \qquad (4.141)$$

so that

$$A = \frac{aL\nu(a)}{aLI_o(La) - 2I_1(La)} . \qquad (4.142)$$

Thus the equation for ν becomes

$$\nu(r) = \nu(a)\ [\frac{aLI_o(Lr) - 2I_1(La)}{I_o(La) - 2I_1(La)}]. \tag{4.143}$$

Our arguments to show that such a motion is impossible are analogous to those made in the case of vertical channel flow. Considering the term $aLI_o(Lr) - 2I_1(La)$ from (4.143), we know that at $r = 0$ this term becomes $aL - 2I_1(La)$. We can show that $aL - 2I_1(La)$ must be less than zero for positive aL. First we note that $I_o(x)$ and $I_1(x)$ are positive functions which increase monotonically as $x \to \infty$.* We may express $I_n(x)$ by the following series expansion:

$$I_n(x) = \sum_{k=o}^{\infty} \frac{(\frac{x}{2})^{2k+n}}{(k!)(k+n)!}, \tag{4.144}$$

where n is an integer. This expansion is valid for all real x. Hence

$$I_1(x) = \sum_{k=o}^{\infty} \frac{(\frac{x}{2})^{2k+1}}{(k!)(k+1)!}. \tag{4.145}$$

As $x \to 0$, $I_1(x) \to \frac{x}{2}$. (4.146)

Then $\qquad x - 2I_1(x) \to 0 \quad \text{as} \quad x \to 0.$ (4.147)

---

*See Lebedev [1972, 1], p. 136.

For $$x > 0, \quad x - 2I_1(x) < 0, \tag{4.148}$$

since $I_1(x) > \frac{x}{2}$ from (4.145). Thus for $aL > 0$,

$$aL - 2I_1(La) < 0. \tag{4.149}$$

At $r = a$, $aL\ I_o(Lr) - 2\ I_1(La) = aL\ I_o(La) - 2\ I_1(La)$. If we can show that $x\ I_o(x) - 2I_1(x) > 0$ for $x > 0$, we will have shown that the term $aL\ I_o(Lr) - 2\ I_1(La)$ experiences a sign change in the interval $0 < r < a$, which is an impossibility because ν is defined so that $0 \leq \nu \leq 1$. We have

$$xI_o(x) = \sum_{k=o}^{\infty} \frac{x(\frac{x}{2})^{2k}}{(k!)^2}, \tag{4.150}$$

but

$$2I_1(x) = \sum_{k=o}^{\infty} \frac{2(\frac{x}{2})^{2k+1}}{(k!)(k+1)!} = \sum_{k=o}^{\infty} \frac{x(\frac{x}{2})^{2k}}{(k!)(k+1)!}. \tag{4.151}$$

Hence

$$xI_o(x) - 2I_1(x) = \sum_{k=o}^{\infty} \frac{x(\frac{x}{2})^{2k}}{k!}\left[\frac{1}{k!} - \frac{1}{(k+1)!}\right] > 0 \tag{4.152}$$

for all $x > 0$, so the motion is impossible, which we expected from our previous result for vertical channel flow.

CHAPTER V

A NONVISCOMETRIC PROBLEM

The General Form of the Solution

We have investigated a number of curvilineal motions. Now let us consider a motion that is neither curvilineal nor, more generally, viscometric. We shall call this motion torsional flow. This flow is curvilineal as it is normally defined. However, we allow here for a dependence of the velocity field on r, so that there is no basis $\underset{\sim}{b}_{<i>}$ for which $\underset{=}{M}$ can be written in the form required for viscometric motions. Since every curvilineal flow is viscometric, this prohibits the motion as we have defined it to be curvilineal as well.

Torsional flow occurs within a region bounded by a vertical cylinder and two rotating horizontal circular plates, as shown in Figure 13.

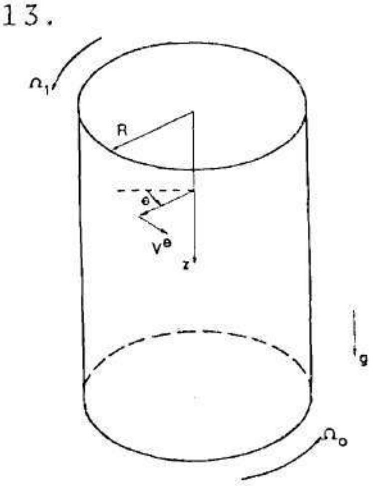

Figure 13. Torsional Flow

The velocity field for this problem takes the form

$$V_z = 0, \quad V_\theta = V_\theta(r,z), \quad \text{and } V_r = 0 \tag{5.1}$$

Since the equation of continuity in this case reduces to $V_\theta \frac{\partial \nu}{r\partial\theta} = 0$, we have

$$\nu = \nu(r,z) \quad \text{only.} \tag{5.2}$$

The "modified" Navier-Stokes equations for this problem are

$$\frac{-\gamma\nu V_\theta^{\,2}}{r} = -2\beta\nu \frac{\partial\nu}{\partial r} + 2\alpha\nu \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\nu}{\partial r}\right) + \frac{\partial^2\nu}{\partial z^2}\right)\right],$$

$$0 = \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) + \frac{\partial^2 V_\theta}{\partial z^2} - \frac{V_\theta}{r^2}\right], \tag{5.3a,b,c}$$

and $$-\gamma\nu g = -2\beta\nu \frac{\partial\nu}{\partial z} + 2\alpha\nu \left[\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\nu}{\partial r}\right) + \frac{\partial^2\nu}{\partial z^2}\right)\right].$$

Now we proceed with the solution to the above equations. Notice that equations (5.3b) and (5.3c) have completely uncoupled. Hence we can solve for V_θ from (5.3b), assuming the separable form

$$V_\theta = \mathcal{R}(r)\mathcal{Z}(z). \tag{5.4}$$

If we put this separable form into equation (5.3b), we have

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\mathcal{R}'(r)\mathcal{Z}(z)\right) + \mathcal{R}(r)\mathcal{Z}''(z) - \frac{\mathcal{R}(r)\mathcal{Z}(z)}{r^2}. \qquad (5.5)$$

This becomes

$$0 = \frac{1}{r}\left(\mathcal{R}'(r)\mathcal{Z}(z) + r\mathcal{R}''(r)\mathcal{Z}(z)\right) + \mathcal{R}(r)\mathcal{Z}''(z) - \frac{\mathcal{R}(r)\mathcal{Z}(z)}{r^2}. \qquad (5.6)$$

Regrouping terms gives

$$0 = \left(\mathcal{R}''(r) + \frac{\mathcal{R}'(r)}{r} - \frac{\mathcal{R}(r)}{r^2}\right)\mathcal{Z}(z) + \mathcal{R}(r)\mathcal{Z}''(z). \qquad (5.7)$$

Dividing through by $\mathcal{R}(r)\mathcal{Z}(z)$ and assuming $\mathcal{R}(r) \neq 0$ and $\mathcal{Z}(z) \neq 0$ yields

$$\left[\left(\mathcal{R}''(r) + \frac{\mathcal{R}'(r)}{r} - \frac{\mathcal{R}(r)}{r^2}\right)/\mathcal{R}(r)\right] = -\left[\mathcal{Z}''(z)/\mathcal{Z}(z)\right] = -k^2. \qquad (5.8)$$

By standard arguments we introduce the separation parameter $-k^2$ and break our solution into the following two equations:

$$\mathcal{Z}''(z) - k^2\mathcal{Z}(z) = 0, \qquad (5.9)$$

and

$$\mathcal{R}''(r) + \frac{\mathcal{R}'(r)}{r} + \left(k^2 - \frac{1}{r^2}\right)\mathcal{R}(r) = 0. \qquad (5.10)$$

For the solution to (5.9) we have

$$\mathcal{Z}(z) = E \sinh kz + F \cosh kz. \tag{5.11}$$

The solution to (5.10) is

$$\mathcal{R}(r) = GJ_1(kr) + HY_1(kr). \tag{5.12}$$

Thus we have obtained a solution for V_θ which is dependent on both r and z.

Now we let

$$\nu(r,z) = R(r)Z(z). \tag{5.13}$$

Then equation (5.3c) becomes

$$-\gamma g = -\beta R(r)Z'(z) + \alpha \frac{\partial}{\partial z} [\frac{1}{r} \frac{\partial}{\partial r} (rZ(z)R'(r) + R(r)Z''(z))] \tag{5.14}$$

after assuming $\nu \neq 0$ and dividing through by ν. We may rewrite this as

$$0 = -\beta R(r)Z'(z) + \alpha \frac{\partial}{\partial z} [Z(z) \frac{1}{r} (R'(r) + rR''(r)) + R(r)Z''(z)], \tag{5.15}$$

where we now solve for the solution to the homogeneous equation in ν by setting $g = 0$. Carrying out the differentiation gives

$$0 = -\beta R(r)Z'(z) + \alpha[(R''(r) + \frac{1}{r} R'(r))Z'(z) + R(r)Z'''(z)] \tag{5.16}$$

as our equation. If we regroup terms in (5.16) we get

$$0 = Z'(z)[-\beta R(r) + \alpha(R''(r) + \frac{1}{r} R'(r))] + \alpha R(r)Z'''(z). \tag{5.17}$$

Now we divide through by $R(r)Z(z)$, assuming $R(r) \neq 0$ and $Z(z) \neq 0$. Then we can introduce the separation parameter λ^2 and rewrite (5.17) as

$$[\beta-\alpha \, (\frac{R''(r) + \frac{1}{r} R'(r)}{R(r)})] = \frac{\alpha z'''(z)}{z'(z)} = \lambda^2. \tag{5.18}$$

We obtain the following two equations for the determination of $R(r)$ and $Z(z)$:

$$R''(r) + \frac{1}{r} R'(r) + - (\frac{\beta-\lambda^2}{\alpha})R(r) = 0 \tag{5.19}$$

and

$$z'''(z) - \frac{\lambda^2}{\alpha} Z'(z) = 0. \tag{5.20}$$

The solutions to these two equations are

$$R(r) = SI_o\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) + TK_o\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) \qquad (5.21)$$

and

$$Z(z) = A \sinh \frac{\lambda}{\sqrt{\alpha}} z + B \cosh \frac{\lambda}{\sqrt{\alpha}} z + c. \qquad (5.22)$$

Writing out fully our solutions for V_θ and ν, we have obtained

$$V_\theta(r,z) = (J_1(Kr) + HY_1(Kr))(E \sinh Kz + F \cosh Kz) \qquad (5.23a)$$

and

$$\nu_H(r,z) = \left(I_o\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) + GK_o\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right)\right)\left(L \sinh \frac{\lambda}{\sqrt{\alpha}} z + M \cosh \frac{\lambda}{\sqrt{\alpha}} z + N\right) \qquad (5.23b)$$

from equations (5.3 b,c).* The solution ν_H is the solution to the homogeneous equation in ν. To obtain the particular solution to (5.14), we try $\nu_p = Dz$. Then we obtain

$$-\gamma g = -\beta D, \quad \text{or} \quad D = \frac{\gamma g}{\beta}\,. \qquad (5.24)$$

Thus the complete expression for ν is

---

*We have combined and rewritten our constants to indicate the correct number of arbitrary constants occurring in our solutions for V_θ and ν_H.

$$\nu(r,z) = (I_o(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\, r) + GK_o\,(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\, r))(L \sinh \frac{\lambda}{\sqrt{\alpha}}\, z$$

$$+ M \cosh \frac{\beta}{\sqrt{\alpha}}\, z + N) + \frac{\gamma_g}{\beta}\, z. \tag{5.25}$$

However, the solutions obtained from (5.3b,c) must also satisfy the coupled equation (5.3a). If we put our solutions for V_θ and ν into (5.3a), we find that our results are incompatible. There are no choices of coefficients for the two functions V_θ and ν determined from (5.3b,c) such that (5.3a) is satisfied.

Once again we are forced to consider an inexact solution by neglecting the inertia term $\frac{-\gamma\nu V_\theta^{\,2}}{r}$ in (5.3a). If we do so, then (5.3a) becomes

$$0 = \beta\,\frac{\partial\nu}{\partial r} + \alpha[\frac{\partial}{\partial r}\,(\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,\frac{\partial\nu}{\partial r}) + \frac{\partial^2\nu}{\partial z^2}]. \tag{5.26}$$

Let us see if we can satisfy (5.26) with our solution from (5.3c) by making a suitable choice of coefficients. We have

$$r\frac{\partial\nu}{\partial r} = (\sqrt{\frac{\beta-\lambda^2}{\alpha}})(-rI_1(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\, r) - GrK_1(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\, r))(L \sinh \frac{\lambda}{\sqrt{\alpha}}\, z +$$

$$M \cosh \frac{\lambda}{\sqrt{\alpha}}\, z + N). \tag{5.27}$$

Therefore

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \nu}{\partial r}\right) = -\left(\frac{\beta-\lambda^2}{\alpha}\right)\left(I_o\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) + GK_o\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right)\right)$$

$$\left(L \sinh \frac{\lambda}{\sqrt{\alpha}} z + M \cosh \frac{\lambda}{\sqrt{\alpha}} z + N\right). \tag{5.28}$$

If we differentiate this expression with respect to r, we obtain

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \nu}{\partial r}\right)\right) = \left(\frac{\beta-\lambda^2}{\alpha}\right)^{3/2}\left(I_1\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) + GK_1\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right)\right)$$

$$\left(L \sinh \frac{\lambda}{\sqrt{\alpha}} z + M \cosh \frac{\lambda}{\sqrt{\alpha}} z + N\right) \tag{5.29}$$

Now

$$\frac{\partial^2 \nu}{\partial z^2} = \left(\frac{\lambda^2}{\alpha}\right)\left(I_o\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) + GK_o\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right)\right)\left(L \sinh \frac{\lambda}{\sqrt{\alpha}} z +\right.$$

$$\left. M \cosh \frac{\lambda}{\sqrt{\alpha}} z\right), \tag{5.30}$$

so that

$$\frac{\partial}{\partial r}\left(\frac{\partial^2 \nu}{\partial z^2}\right) = -\left(\frac{\lambda^2}{\alpha}\right)\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\right)\left(I_1\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) + GK_1\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right)\right)$$

$$\left(L \sinh \frac{\lambda}{\sqrt{\alpha}} z + M \cosh \frac{\lambda}{\sqrt{\alpha}} z\right). \tag{5.31}$$

Equation (5.26) has become

$$0 = (-\beta)\left[-\sqrt{\frac{\beta-\lambda^2}{\alpha}}\left(I_1\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) + GK_1\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right)\right)\left(L \sinh \frac{\lambda}{\sqrt{\alpha}} z + M \cosh \frac{\lambda}{\sqrt{\alpha}} z + N\right)\right] \quad (5.32)$$

$$+ \alpha\left[\left(\frac{\beta-\lambda^2}{\alpha} - \frac{\lambda^2}{\alpha}\right)\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\right)\left(L \sinh \frac{\lambda}{\sqrt{\alpha}} z + M \cosh \frac{\lambda}{\sqrt{\alpha}} z\right)\right] \left(I_1\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right) + GK_1\left(\sqrt{\frac{\beta-\lambda^2}{\alpha}}\,r\right)\right).$$

We can see immediately that we must choose $N = 0$ to satisfy (5.32). We further discover that we must have $0 = 2\beta - 2\lambda^2$, which implies that

$$\lambda = \sqrt{\beta}\ . \quad (5.33)$$

If we put this choice of λ back into our solution for ν we must set $G = 0$. Otherwise, part of the expression becomes unbounded. We obtain

$$\nu(r,z) = (I_o(0))\left(L \sinh \sqrt{\frac{\beta}{\alpha}}\, z + M \cosh \sqrt{\frac{\beta}{\alpha}}\, z\right) + \frac{\gamma_g}{\beta} z. \quad (5.34)$$

Notice the interesting result that the dependence of ν on r has disappeared.

A Specific Boundary Value Problem

If we attempt to formulate a specific torsional problem of this type, we discover that the form of the problem considered must be similar to Figure 14.

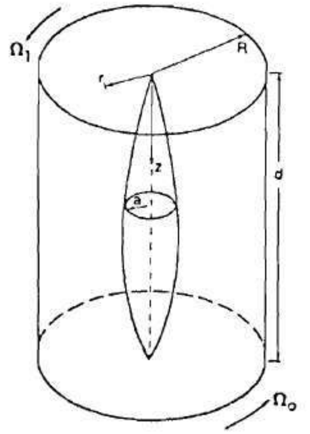

Figure 14. Required Form of the Torsional Flow

We must have the region of equilibrium in the center of the cylinder, because the region of equilibrium can only satisfy the adherence condition at the stationary points on the discs, which are located at their origins. If the region of equilibrium were to come into contact with any other part of the discs, its velocity profile would have to vary linearly in r. This is impossible because the velocity is defined to be constant throughout the regions of equilibrium. The radius at which yielding occurs can in general be any function of z which satisfies the conditions $a(0) = a(d) = 0$. The only case we consider is that for which $a(z) = 0$. This implies that the entire volume of granular material contained between the cylinder and the discs is a region of nonequilibrium. The boundary conditions we apply to the problem are:

$$\lim_{r\to o} V_\theta(r,z) \text{ is finite; } V_\theta(0,r) = \Omega_o r, \quad 0 \leq r \leq R; \tag{5.35a,b,c,d,e,f,g}$$

$$V_\theta(d,r) = \Omega_1 r, \quad 0 \leq r < R; \; V_\theta(r,R) = 0, \; 0 \leq z \leq d;$$

$$\nu(z)\Big|_{z=0} = \nu(0) \quad \text{and} \quad \nu(z)\Big|_{z=d} = \nu(d).$$

To solve this problem, we make use of the fact that our equations under the assumption of negligible inertia are linear and homogeneous. Assuming the necessary convergence, a solution of (5.3b) may be expressed in the form

$$V_\theta(r,z) = \sum_{n=o}^{\infty} J_1(k_n r)(E_n \sinh k_n z + F_n \cosh k_n z), \tag{5.36}$$

where the k_n are parameters to be determined by the boundary condition (5.35c).

Notice that along the outer edges of the discs the conditions of adherence lead to contradictory requirements on the velocity field. On the one hand, if we consider the granular material that is in contact with the surface of the upper disc, to prevent slipping it must have velocity

$V_\theta = \Omega_1 R$ at $r = R$. On the other hand, granular material in contact with the surface of the stationary cylinder must be stationary, including the material which is in contact at $r = R$, $z = 0$ and $z = d$. Obviously, both sets of conditions cannot be satisfied at once. We could attempt to correct this deficiency by taking away the cylinder and assuming a constant hydrostatic atmospheric pressure acting on the material between the two discs. Here we assume that if we put the discs close together the material will stay in the gap between the discs. (See Lodge [1964, 1], p. 198, for a discussion of this point.) However, this assumption leads to other difficulties in that the surface in contact with the air will not in general be cylindrical, so that the condition the stress must satisfy is highly complicated. We choose to retain our boundary condition as stated despite its contradictory nature. In doing so, we argue that the velocity profile for the granular material at $z = 0$ or at $z = d$ will be as shown in Figure 15.

The velocity profile should retain its linear form until it nears the cylindrical wall. Then it will probably taper down in some unknown fashion until it becomes zero again at the wall. Notice that one or both of the no-slip conditions must be violated. The velocity under these conditions may be unsteady. However, whatever form it may take, we assume that its deviation from the ideal form shown in Figure 15 is small enough so that our boundary conditions

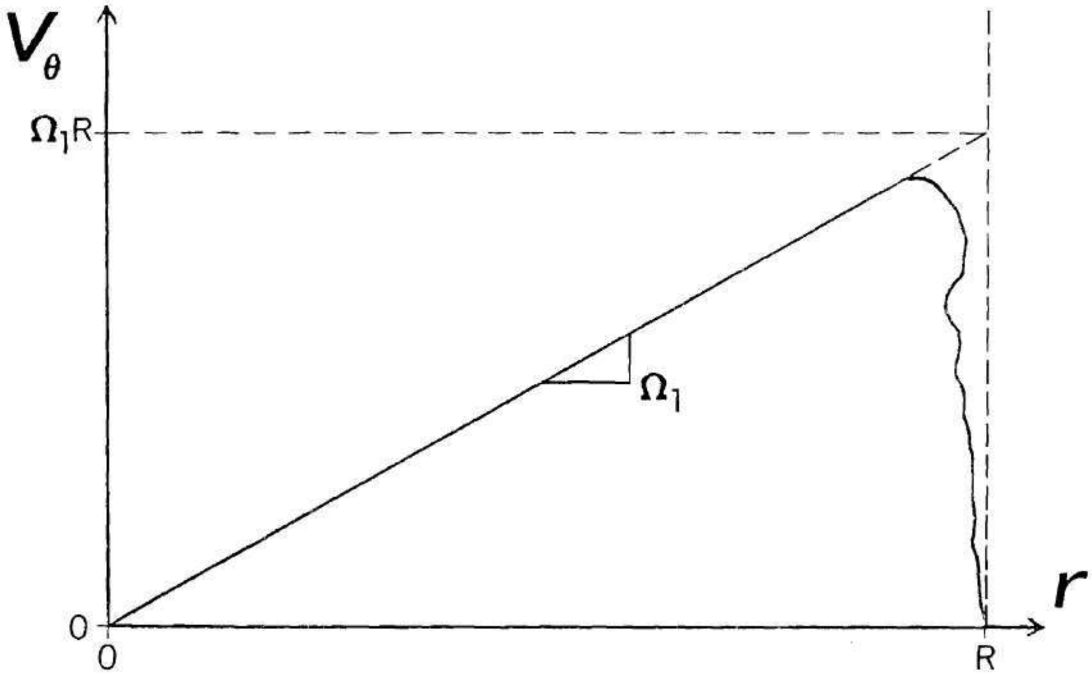

Figure 15. The Assumed Velocity Profile

as given will yield reasonably good results. It is not known whether this assumption is justified. If this procedure yields results that are close to those obtained from experiment, then it would appear that our assumptions are valid. Discrepancies between results obtained here and experimental results may arise from one of the following three factors: (1) second order effects are important for the type of granular material used in the experiment, so that the linear theory is insufficient to describe the motion, (2) the inertia term is too large to be neglected, and (3) the deviation of the velocity profile near the corners between the cylinder wall and disc edge from the ideal case is large enough to render our boundary conditions invalid. We proceed with the solution, bearing in mind these possible pitfalls.

First we have

$$V_\theta(z,R) = 0 = \sum_{n=0}^{\infty} J_1(k_nR)(E_n \sinh k_nz + F_n \cosh k_nz). \quad (5.37)$$

Therefore

$$J_1(k_nR) = 0, \quad \text{or} \quad k_nR = x_{1n}, \quad (5,38a,b)$$

where x_{1n} are the zeroes of $J_1(z)$.* Now we must have

*Since $J_1(0) = 0$, we have $x_{10} = 0$. However, we must discard this zero of J_1, since we are seeking nontrivial eigenfunctions. Henceforth we sum from $n = 1$ to $n = \infty$.

$$V_\theta(0,r) = \Omega_o r = \sum_{n=1}^{\infty} J_1 \left(\frac{x_{1n}}{R} r\right) F_n. \tag{5.39}$$

Using the standard arguments and orthogonality conditions for Bessel functions as outlined in Lebedev, [1972, 1] pp. 128-9, we find that

$$F_m = \frac{2}{R^2 J_2{}^2(x_{1m})} \int_o^R \Omega_o r^2 J_1 \left(x_{1m} \frac{r}{n}\right) dr. \tag{5.40}$$

Let

$$r = \frac{Ry}{x_{1m}} . \tag{5.41}$$

Then

$$F_m = \frac{2R^3 \Omega_o}{R^2 x_{1m}^3 \; J_2{}^2(x_{1m})} \int_o^{x_{1m}} y^2 J_1(y)\, dy. \tag{5.42}$$

We can use the identity

$$y^2 J_1(y) = \frac{d}{dy} [y^2 J_2(y)] \tag{5.43}$$

to obtain

$$F_m = \frac{2R\Omega_o}{x_{1m}^3 \; J_2{}^2 \; (x_{1m})} [y^2 J_2(y)] \Big|_o^{x_{1m}} \tag{5.44}$$

So

$$F_m = \frac{2R\Omega_o}{x_{1m} J_2 (x_{1m})} . \tag{5.45}$$

Now we must have

$$V_\theta(d,r) = \Omega_1 r = \sum_{n=1}^{\infty} J_1(k_n r)(E_n \sinh k_n d + F_n \cosh k_n d). \tag{5.46}$$

Again we use the standard arguments to show that

$$E_m \sinh \left(\frac{x_{1m} d}{R}\right) + F_m \cosh \left(\frac{x_{1m} d}{R}\right) = \frac{2\Omega_1}{R^2 J_2^{\,2}(x_{1m})} \int_o^R r^2 J_1(x_{1m} \tfrac{r}{R}) dr. \tag{5.47}$$

By a similar substitution as for our previous boundary condition, we can show that

$$\frac{2\Omega_1}{R^2 J_2^{\,2}(x_{1m})} \int_o^R r^2 J_1(x_{1m} \tfrac{r}{R}) dr = \frac{2R\Omega_1}{x_{1m} J_2 (x_{1m})} . \tag{5.48}$$

Finally we have

$$E_m = \frac{2R(\Omega_1 - \Omega_o \cosh (\frac{x_{1m}}{R} d))}{x_{1m} J_2 (x_{1m}) \sinh (\frac{x_{1m}}{R} d)} . \tag{5.49}$$

Therefore the form of the velocity for this problem is

$$V_\theta(r,z) = \sum_{n=1}^{\infty} \frac{2R}{x_{1n}\, J_2\,(x_{1n})} \{J_1(\frac{x_{1n}}{R} r)[(\Omega_1 - \Omega_o \cosh(\frac{x_{1n}}{R} d))$$
$0<r<R$
$0<z<d$

$$\frac{\sinh(\frac{x_{1n}}{R} z)}{\sinh(\frac{x_{1n} d}{R})} + \Omega_o \cosh(\frac{x_{1n}}{R} z)]\}. \qquad (5.50)$$

Earlier we found that the general form for ν in torsional flow is

$$\nu(z) = A \sinh \sqrt{\frac{\beta}{\alpha}}\, z + B \cosh \sqrt{\frac{\beta}{\alpha}}\, z + \frac{\gamma_g}{\beta} z. \qquad (5.51)$$

If we apply one of our boundary conditions, we have

$$\nu(z)\Big|_{z=o} = \nu(0) = B. \qquad (5.52)$$

If we apply the second boundary condition, we obtain

$$\nu(z)\Big|_{z=d} = \nu(d) = A \sinh \sqrt{\frac{\beta}{\alpha}}\, d + \nu(0) \cosh \sqrt{\frac{\beta}{\alpha}}\, d + \frac{\gamma_g}{\beta} d. \qquad (5.53)$$

Thus the solution for ν is

$$\nu(z) = \frac{\nu(0)}{\sinh(\sqrt{\frac{\beta}{\alpha}}\, z)} [(\frac{\nu(d) - \frac{\gamma_g d}{\beta}}{\nu(0)} - \cosh(\sqrt{\frac{\beta}{\alpha}}\, d))\sinh(\sqrt{\frac{\beta}{\alpha}}\, z) +$$
$0<z<d$
$0<r<R$

$$\sinh(\sqrt{\frac{\beta}{\alpha}}\, d)\cosh \sqrt{\frac{\beta}{\alpha}}\, z] + \frac{\gamma_g z}{\beta} \qquad (5.54)$$

The stress in the regions of nonequilibrium for this problem is given by

$$\underline{\underline{T}} = (\beta_o - \beta\nu^2 + \alpha(\frac{d\nu}{dz})^2 + 2\alpha\nu\frac{d^2\nu}{dz^2}) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 2\alpha \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{d\nu}{dz})^2 \end{vmatrix}$$

$$+ 2\mu \begin{vmatrix} 0 & (\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r}) & 0 \\ (\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r}) & 0 & (\frac{\partial V_\theta}{\partial z}) \\ 0 & (\frac{\partial V_\theta}{\partial z}) & 0 \end{vmatrix} . \qquad (5.55)$$

Since V_θ and ν are now known, the stress tensor can be calculated from the above. Note that in general T_{rz} is the only stress component that is zero.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

We have considered the problem of the motion of granular materials as represented by the linear constitutive equations. General results valid for all viscometric and curvilineal motions of granular materials have been obtained. We have considered several specific boundary value problems of curvilineal motions. In some cases, exact solutions were obtained. For several of the flows considered we were forced to assume that the inertia term appearing in the "modified" Navier-Stokes equations is negligible to obtain an inexact solution.

Several observations may be made at this point. We showed for both vertical channel flow and Poiseuille flow that symmetric motions with central regions of nonequilibrium are impossible. There is no obvious physical reason to reject such motions. However, we find that the boundary conditions cannot be physically satisfied by Coulomb granular materials.

We have found that, in general, by making certain assumptions about the spatial dependence of $\underset{\sim}{v}$ and ν, we are able to reduce the system of partial differential equations describing the motions of the grains in the regions of nonequilibrium to a set of ordinary differential equations. Some of the resulting equations are uncoupled, but in several instances there is at least one coupled equation we must contend with. This coupling in some instances prevents us from obtaining exact solutions. For these cases, we find no exact solutions and we proceed to obtain inexact solutions. We argue that under the proper restrictions an inexact solution is preferable to no solution. We must be cautious in our application of these inexact solutions, for as we showed in Figure 7 of Chapter IV, the inexact and exact solution are not similar for large angular velocities in the case of flow between rotating concentric cylinders. In a positive light, Figure 7 also shows that the inexact solution yields results close to the exact case for "slow" flows. We can with some degree of confidence extend this argument to include our other inexact solutions.

In our investigation of torsional flow, we have found a difficulty in matching no-slip boundary conditions at the edges of the disc. The no-slip condition cannot be valid for this problem because it leads to mutually contradictory requirements on the velocity field at the edges of the discs. We have made our own assumptions regarding the velocity profiles near the edges of the discs as shown in Figure 16 of Chapter V. The accuracy of our result depends to some extent on the validity of this assumption.

Finally, we have determined conditions for the steady universal motions of Coulomb granular materials.* It is found that steady universal motions of Navier-Stokes fluids can also be steady universal solutions in the case of the motions of Coulomb granular materials if two additional partial differential equations which are coupled between $\underset{\sim}{v}$ and ν are satisfied. The types of universal solutions obtained in the case of Navier-Stokes fluids have been investigated by Marris [1971, 2], [1972, 3], and [1975, 2]. Imposing two additional conditions on the velocity field reduces the variety of possible solutions even further in the case of granular materials.

Recommendations

The theory proposed by Goodman and Cowin introduces several material parameters to describe the behavior of granular materials. These parameters can in general be determined for particular granular materials by experiment. No specific technique is proposed for determining these parameters. However, since universal motions of granular materials are flows which are independent of the material properties as represented in the "modified" Navier-Stokes equations, they should be valuable for obtaining accurate measurements of the material properties.

We have determined the analytical solutions (both

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*See Appendix A.

exact and inexact) to a number of boundary value flow problems based on the theory proposed by Goodman and Cowin. Some of the flows considered in this work should be reproducible experimentally. The validity of the theory and of the assumptions made in the cases for which we obtained only inexact solutions can thus be tested by experiment.

In this work we have considered only steady motions of Coulomb granular materials. No unsteady solutions have been obtained using the theory of Goodman and Cowin. Unsteady solutions would be valuable in predicting the motion occurring after a distributed volume is released from some initial static configuration for various specific types of boundary value problems.

The mathematical methods available for the solutions of the types of problems considered in this work are somewhat limited at the present time. Other techniques need to be devised before exact solutions to many of the viscometric motions can be found. Unfortunately, there are a large number of flow problems which fall into this category of problems whose exact solutions have not yet been obtained. This is true for the case of granular materials as well as for other new continuum models for various types of materials, including simple fluids.* Until other means are at our disposal for obtaining the exact solutions, we must be content

*See Truesdell, C. and W. Noll [1965, 1, §§ 111-115] for a survey of special flow problems of simple fluids for which both exact and inexact solutions are presented.

with the limited types of solutions which have been found to date.

APPENDICES

APPENDIX A

STEADY UNIVERSAL MOTIONS OF GRANULAR MATERIALS

The motions of cohesionless granular materials in the regions of nonequilibrium are characterized by the continuity equation

$$\dot{\nu} + \nu \text{ div } \underline{v} = 0, \qquad (A.1)$$

and by the dynamical equation

$$\gamma\nu\dot{\underline{v}} = \gamma\nu\underline{b} - 2\beta\nu \text{ grad } \nu + 2\alpha\nu \text{ grad (div (grad } \nu))$$

$$+ (\lambda+\mu) \text{ grad (div } \underline{v}) + \mu \text{ div (grad } \underline{v}). \qquad (A.2)$$

The continuity equation may be rewritten as

$$\frac{\partial\nu}{\partial t} + \underline{v} \cdot \text{grad } \nu + \nu \text{ div } \underline{v} = 0,$$

and the dynamical equation as

$$\gamma\nu \left(\frac{\partial\underline{v}}{\partial t} + \underline{v} \cdot \text{grad } \underline{v}\right) = \gamma\nu\underline{b} - 2\beta\nu \text{ grad } \nu + 2\alpha\nu \text{ grad (div (grad } \nu))$$

$$+ (\lambda+\mu) \text{ grad (div } \underline{v}) + \mu \text{ div (grad } \underline{v}). \qquad (A.3)$$

Let us consider steady isochoric motions. This implies that

$$\operatorname{div} \underline{v} = 0, \quad \frac{\partial \nu}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \underline{v}}{\partial t} = \underline{0}. \tag{A.4}$$

The continuity equation reduces to

$$\underline{v} \cdot \operatorname{grad} \nu = 0, \tag{A.5}$$

and the dynamical equation becomes

$$\gamma\nu\underline{v} \cdot \operatorname{grad} \underline{v} = \gamma\nu\underline{b} - 2\beta\nu \operatorname{grad} \nu + 2\alpha\nu \operatorname{grad} (\operatorname{div} (\operatorname{grad} \nu))$$

$$+ \mu \operatorname{div} (\operatorname{grad} \underline{v}). \tag{A.6}$$

Now considering only conservative body forces so that

$$\underline{b} = -\operatorname{grad} V,$$

and using the well known identities

$$\underline{v} \cdot \operatorname{grad} \underline{v} = \underline{\omega} x \underline{v} + \operatorname{grad} \frac{v^2}{2} \tag{A.7}$$

and

$$\operatorname{div} (\operatorname{grad} \underline{v}) = \operatorname{grad} (\operatorname{div} \underline{v}) - \operatorname{curl} \operatorname{curl} \underline{v}, \tag{A.8}$$

we obtain

$$\gamma\nu(\underset{-}{\omega}x\underset{-}{v} + \text{grad}\ \frac{v^2}{2}) = -\ \gamma\nu\ \text{grad}\ V - 2\beta\nu\ \text{grad}\ \nu$$

$$+\ 2\alpha\nu\ \text{grad}\ (\text{div}\ (\text{grad}\ \nu)) - \mu\ \text{curl}\ (\text{curl}\ \underset{-}{v}) \qquad (A.9)$$

as our dynamical equation.

We divide throughout (A.9) by $\gamma\nu$, assuming $\nu \neq 0$ in the regions of nonequilibrium. We obtain

$$\underset{-}{\omega}x\underset{-}{v} + \text{grad}\ \frac{v^2}{2} = -\ \text{grad}\ V - \frac{2\beta}{\gamma}\ \text{grad}\ \nu$$

$$+\ 2\ \frac{\alpha}{\gamma}\ \text{grad}\ (\text{div}\ (\text{grad}\ \nu)) - \frac{\mu}{\gamma\nu}\ \text{curl}\ (\text{curl}\ \underset{-}{v}). \qquad (A.10)$$

We are seeking universal solutions of the dynamical equation, i.e., motions independent of the material properties μ, α, and β. We take the curl of both sides of (A.10) to obtain

$$\text{curl}\ (\underset{-}{\omega}x\underset{-}{v}) = \frac{-\mu}{\gamma}\ (\text{curl}\ [\frac{1}{\nu}\ \text{curl}\ (\text{curl}\ \underset{-}{v})]). \qquad (A.11)$$

We are assuming that μ, α, and β are all constant material coefficients. For universal motions we must have

$$\text{curl}\ (\underset{-}{\omega}x\underset{-}{v}) = 0 \qquad (A.12)$$

and

$$\text{curl } [\frac{1}{\nu} \text{ curl } (\text{curl } \underline{v})] = 0. \tag{A.13}$$

Therefore the set of conditions any universal solutions of a cohesionless granular material must satisfy in the regions of nonequilibrium are

$$\text{div } \underline{v} = 0, \tag{A.14}$$

$$\underline{v} \cdot \text{grad } \nu = 0, \tag{A.15}$$

$$\text{curl } (\underline{\omega} \text{x} \underline{v}) = 0, \tag{A.16}$$

and

$$\text{curl } [\frac{1}{\nu} \text{ curl } (\text{curl } \underline{v})] = 0. \tag{A.17}$$

Comparing these conditions with those required of a Navier-Stokes fluid, we note that for Navier-Stokes fluids ν = a constant, so that our conditions (A.14) through (A.17) reduce to

$$\text{div } \underline{v} = 0, \tag{A.18}$$

$$\text{curl } (\underline{\omega} \text{x} \underline{v}) = 0, \tag{A.19}$$

and

$$\text{curl}\ [\text{curl}\ (\text{curl}\ \underset{\sim}{v})] = 0\ , \tag{A.20}$$

which are the familiar conditions for universal solutions of a Navier-Stokes fluid.*

Equation (A.17) may be expanded to obtain

$$\text{grad}\ \frac{1}{\nu}\ \text{x curl}\ (\text{curl}\ \underset{\sim}{v}) + \frac{1}{\nu}\ \text{curl}\ [\text{curl}\ (\text{curl}\ \underset{\sim}{v})] = 0. \tag{A.20}$$

Suppose we have a steady velocity field $\hat{\underset{\sim}{v}}$ which is a universal solution of the Navier-Stokes equation, so that

$$\text{div}\ \hat{\underset{\sim}{v}} = 0, \tag{A.21}$$

$$\text{curl}\ (\hat{\underset{\sim}{\omega}} \text{x} \hat{\underset{\sim}{v}}) = 0, \tag{A.22}$$

and

$$\text{curl}\ [\text{curl}\ (\text{curl}\ \hat{\underset{\sim}{v}})] = 0. \tag{A.23}$$

We want to seek solutions $\hat{\underset{\sim}{v}}$ for the case of granular materials which are also solutions of the Navier-Stokes equation. The new conditions required of $\hat{\underset{\sim}{v}}$ (which already satisfies the

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*See A. W. Marris, [1975, 2].

conditions for universal motion required for Navier-Stokes fluids) and ν are that

$$\underline{\hat{v}} \cdot \text{grad } \nu = 0, \tag{A.24}$$

and

$$\text{grad } \frac{1}{\nu} \times \text{curl } (\text{curl } \underline{\hat{v}}) = 0. \tag{A.25}$$

In the set of all universal solutions of the Navier-Stokes equation there is a subset $\underline{\hat{\hat{v}}}$ of solutions which satisfy a condition on $\underline{\hat{\hat{v}}}$ that is stronger than required. These are the solutions which satisfy

$$\text{curl } (\text{curl } \underline{\hat{\hat{v}}}) = 0. \tag{A.26}$$

Since for isochoric motions

$$\nabla^2 \underline{\hat{\hat{v}}} = - \text{curl } (\text{curl } \underline{\hat{\hat{v}}}), \tag{A.27}$$

it may be immediately recognized that this subset of solutions consists of harmonic functions. If we seek universal solutions for granular materials whose velocity field satisfies this stronger condition, then the condition that

$$\text{grad } (\tfrac{1}{\nu}) \times \text{curl } (\text{curl } \underline{\hat{\hat{v}}}) = 0 \tag{A.28}$$

is always satisfied, and we are left with the single condition that

$$\hat{\hat{\underline{v}}} \cdot \text{grad } \nu = 0. \qquad \text{(A.29)}$$

We may infer from this condition that on each streamline, ν will be determined to within an arbitrary constant.

APPENDIX B

THE DEFINITION OF THE EQUILIBRIUM AND NONEQUILIBRIUM REGIONS

In this work we are primarily interested in determining the velocity fields and functional representation of the volume distribution function in the regions of nonequilibrium of granular materials. For this reason, instead of using the constitutive equation for the region defined by Goodman and Cowin [1971, 1] as a part of the problem solution, we have instead employed a dynamic yield condition to serve as a boundary condition* for the region of nonequilibrium. Essentially, we argue that at the surfaces of transition from regions of equilibrium to regions of nonequilibrium, we must satisfy the requirement that $\underline{\underline{D}}$ at the surface equal zero. This condition provides a boundary condition to be satisfied by the velocity field. If the surface of transition is of a form such that it can be characterized by a single constant parameter (the radius of a cylinder or the width of a region), then this requirement that $\underline{\underline{D}} = \underline{\underline{0}}$ generally provides a condition for the determination of the

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*Jenkins [1975, 1] has obtained two boundary conditions to be satisfied by $\underline{T}$ and $\underline{h}$ for the regions of equilibrium. The relationship of these conditions to the condition on $\underline{\underline{D}}$ employed in this work has not yet been determined.

parameter. The further requirement that the velocity field be constant in a region of equilibrium provides us with an additional boundary condition on the velocity field.

APPENDIX C

THE NONUNIQUENESS OF THE SOLUTIONS

In their solution to the vertical channel flow of a Coulomb granular material, Goodman and Cowin assumed that a central plug (being a region of equilibrium) would form, and that the velocity and volume distribution function would be symmetric about the central axis. The assumption of a central plug was apparently made on the basis of the findings of previous experimental work in granular flow. While their results are valid and appear to correlate well with those findings, there is nothing in the present state of the theory of granular materials to rule out a number of other possible flow types for the same problem. Without drawing on any apparent experimental justification, we can postulate a flow with a region of nonequilibrium in the center of the channel and regions of equilibrium near the channel walls. We considered symmetric flows of this type in Chapter IV and showed that such flows are impossible.

Other alternative flows are possible for several of the problems we have considered. For example, we can take the inexact solution we have obtained for flow between concentric cylinders and construct a more complex problem with alternating regions of equilibrium and nonequilibrium as shown in Figure 16. The inner face of the outer region of

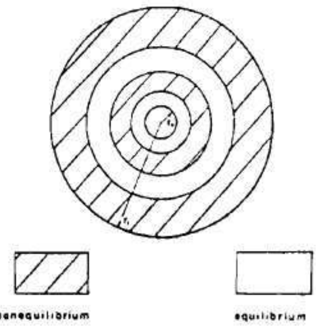

Figure 16. Alternating Regions Between Concentric Cylinders

equilibrium is equivalent to the inner face of the outer cylinder of the problem we worked in Chapter IV with our exact solution. The outer face represents the surface of the inner cylinder for a problem identical to the one we worked in Chapter IV in which the region of equilibrium experienced rigid body rotation only. We can thus tie together an indefinite number of alternating bands of regions of equilibrium and nonequilibrium. We have a nonuniqueness in our solution because any arbitrary number of regions of this type are possible between the two cylinders.

The vertical channel flow and vertical flow between concentric cylinders also exhibit a nonuniqueness in that we can construct solutions of the type shown in Figure 17. We have no reason based on the theory to expect one solution to be more likely than any other.

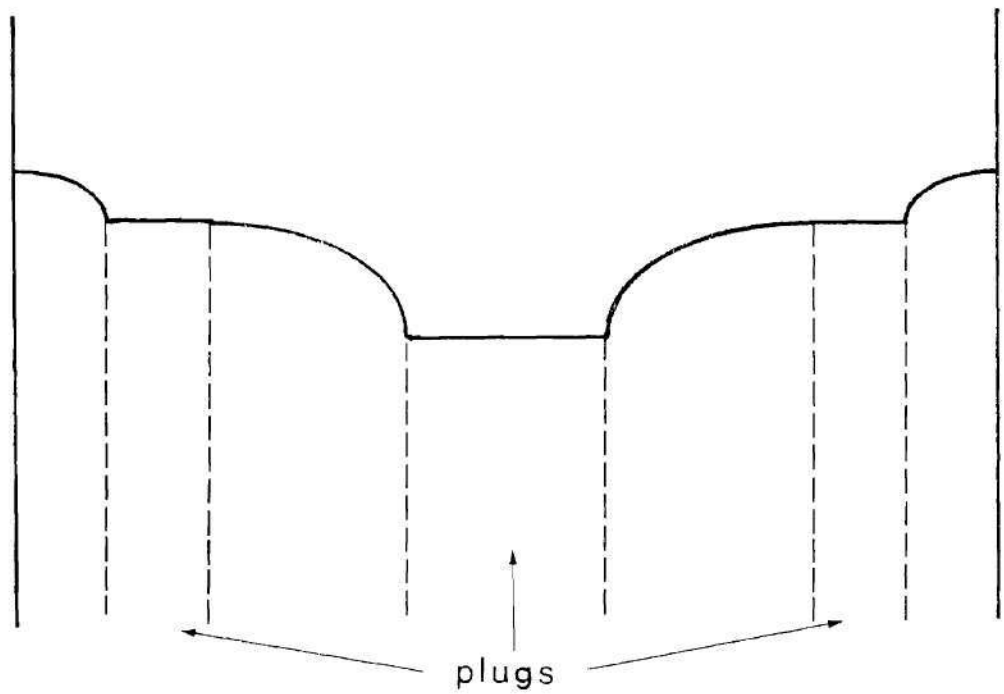

Figure 17. Multiple Plug Channel Flow

Let us see if we can account for the reasons we have this nonuniqueness in our solutions. First, our solutions are not analytically defined solutions over the entire body of granular material, as shown in Figure 18. We note that

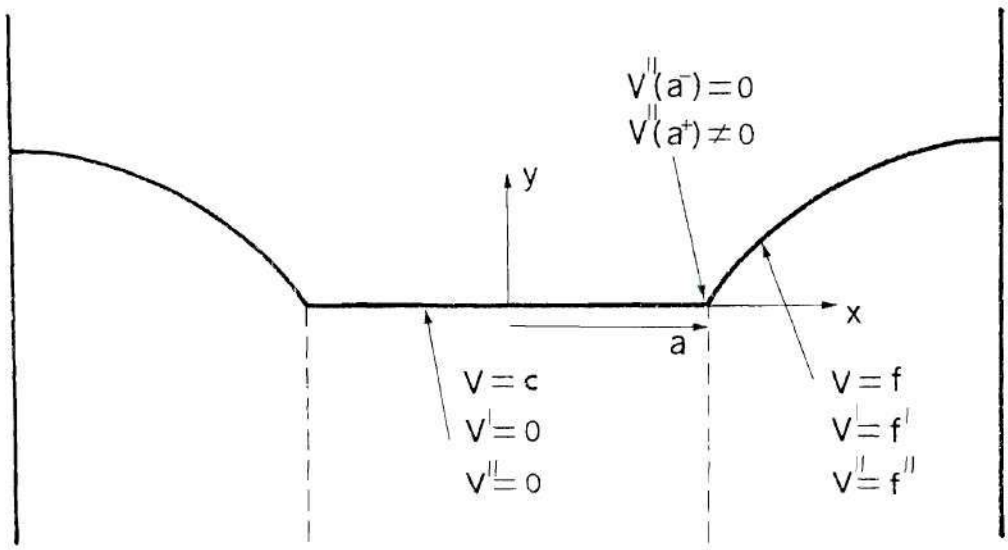

Figure 18. The Nonanalytic Form of the Solution

although $V(x)$ and $V'(x)$ are continuous, $V''(x)$ is discontinuous at some points over the interval. The fact that we have defined that conditions describing $V(x)$ such that the second derivative is in general discontinuous at some point or points is evidently one reason for our multitude of possible solutions. It allows us to tie together pieces of horizontal lines (in the case shown in Figure 19) which satisfy conditions in the equilibrium regions with compatible curves of nonzero second derivative which satisfy conditions in the nonequilibrium regions.

A second reason we might obtain nonunique solutions is that the dynamical and continuity equations derived from our constitutive equation are in general nonlinear and coupled in the two variables $\underline{v}$ and ν. However, in the cases we considered, we managed to reduce the equations to a set of ordinary equations which were coupled in $\underline{v}$ and ν in at most one equation. For more general cases there is no guarantee that solutions will be unique. The consideration of whether this type of nonuniqueness enters into a problem depends on the specific nature of the problem and any assumptions that have to be made in order to obtain a solution.

These are the two unrelated factors which may imply a nonuniqueness of our solutions. Either or both of them may apply to the solution of a particular problem. Thus there is no reason to believe that an infinite number of possible solutions to a particular problem will not exist.

However, we have solved a fully developed, steady flow problem ignoring the fact that for any real flow there is an initial configuration. Some inducement was provided to set the material in motion, whereupon it underwent some transient unsteady motion before it attained its steady state. In reaching its steady state, it may be that there is some preferred steady motion, depending on the initial configuration. Our multitude of solutions may in fact represent steady state solutions for different initial configurations.

It is also possible that some solutions are more stable than others, as is the case with the buckling of elastic beams. There may be a single solution that is stable and a number of quasi-stable solutions which will reorient to the one stable solution given the slightest perturbation. It would seem that for given initial conditions a flow will seek the form with minimum potential energy. However, this is all conjecture. What we need is a stability analysis of the various steady state solutions to see which is most stable. We would also benefit from having solutions to some unsteady problems with a given initial configuration. Only then will we be able to predict with some confidence what solutions are most likely to be manifested physically by granular materials.

APPENDIX D

THE STATEMENT OF THE BALANCE OF EQUILIBRATED FORCE

Recall from (1.11) that we have the following field equation representing the balance of equilibrated force:

$$\gamma \nu k \ddot{\nu} = \operatorname{div} \underset{\sim}{h} + \gamma \nu \ (\ell + g). \tag{D.1}$$

We shall investigate the above equation for the case of the linear theory of Coulomb granular materials. For the linear theory, we have

$$\underset{\sim}{h} = 2\alpha \operatorname{grad} \nu \tag{D.2}$$

and

$$g - g^{o} = -\zeta \dot{\nu} + -\delta \operatorname{tr} \underset{\approx}{D}, \tag{D.3}$$

where α, ζ and δ are scalar functions of ν_o, ν, grad ν, γ and θ in general.

Let us look at the case when α = constant. Then

$$\operatorname{div} \underset{\sim}{h} = 2\alpha \operatorname{div} \operatorname{grad} \nu, \tag{D.4}$$

and

$$g = g^{o} - \zeta\dot{\nu} - \delta \mathrm{tr}\, \underline{\underline{D}}. \qquad \text{(D.5)}$$

We are considering only isochoric motions, which implies that $\mathrm{tr}\underline{\underline{D}} = 0$. Furthermore, under the condition of isochoric motions the continuity equation becomes simply

$$\dot{\nu} = 0. \qquad \text{(D.6)}$$

Hence (D.5) reduces to

$$g = g^{o}. \qquad \text{(D.7)}$$

From Goodman and Cowin [1972, 1], the equilibrium value of the intrinsic equilibrated body force is

$$g^{o} = \frac{p-\hat{p}}{\gamma\nu^2}, \qquad \text{(D.8)}$$

where

$$p = \gamma^2\nu\,\frac{\partial\psi}{\partial\gamma}, \quad \hat{p} = \gamma\nu^2\,\frac{\partial\psi}{\partial\nu}, \qquad \text{(D.9a,b)}$$

and ψ is the <u>free energy function</u>, given by $\psi = \varepsilon - \theta\eta$.

Our force equilibrium equation may now be written as

$$\gamma\nu k\ddot{\nu} = 2\alpha \text{ div grad } \nu + \gamma\nu \left(\ell + \frac{1}{\nu}\frac{\partial\psi}{\partial\gamma} - \frac{\partial\psi}{\partial\nu}\right). \qquad \text{(D.10)}$$

But $\dot{\nu} = 0$ implies that $\ddot{\nu} = 0$, so the equation reduces to

$$0 = 2\alpha \text{ div grad } \nu + \gamma\nu \left(\ell + \frac{1}{\nu}\frac{\partial\psi}{\partial\gamma} - \frac{\partial\psi}{\partial\nu}\right). \qquad \text{(D.11)}$$

This can be written as

$$\frac{2\alpha \text{ div grad } \nu}{\gamma\nu} + \ell = \frac{\partial\psi}{\partial\nu} - \frac{1}{\nu}\frac{\partial\psi}{\partial\gamma}. \qquad \text{(D.12)}$$

Thus in the special case of isochoric motion the equation of equilibrated force reduces to an equation serving to define a required relationship between ν, ℓ, ψ, γ and α.

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